A Law of Large Numbers approach to valuation in life insurance

Tom Fischer*
Heriot-Watt University, Edinburgh

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Abstract

The classical Principle of Equivalence ensures that a life insurance company can accomplish that the mean balance per policy converges to zero almost surely for an increasing number of independent policyholders. By certain assumptions, this idea is adapted to the general case with stochastic financial markets. The implied minimum fair price of general life insurance policies is then uniquely determined by the product of the assumed unique equivalent martingale measure of the financial market with the physical measure for the biometric risks. The approach is compared with existing related results. Numeric examples are given.

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1 Introduction

Roughly speaking, the Principle of Equivalence of traditional life insurance mathematics states that premiums should be calculated such that incomes and losses are "balanced in the mean". Under the assumption that financial markets are deterministic, this idea leads to a valuation method usually called

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"Expectation Principle". The use of the two principles ensures that a life office can accomplish that (i.e. can buy hedges such that) the mean balance per policy converges to zero almost surely for an increasing number of policyholders. This is often confered to as the ability to "diversify" mortality (or biometric) risks. The main mathematical ingredients for this diversification are the stochastic independence of individual lives and the Strong Law of Large Numbers (SLLN). To obtain the mentioned convergence, it is neither necessary to have identical policies, nor to have i.i.d. lifetimes.

In modern life insurance mathematics, where financial markets are assumed to be stochastic and where more general products (e.g. unit-linked ones) are taken into consideration, the widely accepted valuation principle is an expectation principle, too. However, the respective probability measure is different since the minimum fair price or market value of an insurance claim is determined by the no-arbitrage pricing method known from financial mathematics. The respective equivalent martingale measure (EMM) is the product of the given EMM of the financial market with the physical measure for the biometric risks. Throughout the paper, we will call this kind of valuation the *product measure principle*. Although the result is not as straightforward as in the traditional case, a convergence property similar to the one mentioned above can be shown. So, diversification of biometric risks is still possible in the presence of stochastic financial markets, where payments related to e.g. unit-linked life policies of different policyholders may not be independent.

The aim of this paper is the derivation of an equivalent martingale measure for the pricing of life insurance policies starting from the assumption that, under the induced valuation principle, diversification of biometric risks should be possible by means of a convergence property as above, i.e. a life insurance company should be able to accomplish that the mean balance per contract converges to zero almost surely for an increasing number of independent policyholders. We will see that, under certain assumptions, the EMM then is uniquely determined and given by the product measure mentioned earlier, i.e. by the product of the given EMM of the financial market with the physical measure for the biometric risks. In different versions, diversification approaches have appeared in the literature on valuation. Considered as somehow straightforward, they are usually stated without proofs and for identical policies and i.i.d. lives, only. However, the derivation of a unique equivalent martingale *measure* and respective convergence properties for *varying* types of policies and lives at the same time, as carried out in this paper, needs a formally different setup and different mathematical tools than the derivation of a unique *pricing rule* for infinitely many *identical* policies for i.i.d. lives, as done in some papers. In this sense, the present paper has a technical focus. Particular emphasis is put on mathematically rigorous and explicit model assumptions necessary for the derivation of the mentioned results. For instance, we state integrability conditions for cash flows of not necessarily identical policies that are sufficient for the application of the SLLN even if independence gets lost by common financial risks.

Research on the valuation of unit-linked life insurance products already started in the late 1960s. One of the first results that was in its core identical to the product measure principle was Brennan and Schwartz (1976). In this paper, the authors ”eliminate mortality risk” by assuming an “average purchaser of a policy”, which clearly is a diversification argument. More recent papers mainly dedicated to valuation following this approach are Aase and Persson (1994) for the Black-Scholes model and Persson (1998) for a stochastic interest rate model. Aase and Persson (1994), but also other authors, a priori suppose independence of financial and biometric events. In their paper, an arbitrage-free and complete financial market ensures the uniqueness of the financial EMM. The product measure principle is here motivated by a diversification argument, but also by “risk-neutrality” of the insurer with respect to biometric risks (cf. Aase and Persson (1994), Persson (1998)). A more detailed history of valuation in (life) insurance can be found in Møller (2002), see also the references therein.

There exist other derivations of the product measure principle which do not rely on diversification arguments. In Møller (2001), for example, the product measure coincides with the so-called minimal martingale measure (cf. Schweizer, 1995b). The works Møller (2002, 2003a, 2003b) also consider valuation, but focus on hedging (mainly quadratic criteria), respectively advanced premium principles. Becherer (2003) uses exponential utility functions to derive prices of contracts. In an example for a certain type of contract for i.i.d. lives, he shows that the product measure principle evolves in the limit for infinitely many policyholders. In general, no-arbitrage pricing of insurance cash flows using martingales and equivalent martingale measures, was introduced by Delbaen and Haezendonck (1989) and Sondermann (1991). Later, Steffensen (2000) described possible sets of price operators for life insurance contracts by respective sets of eqivalent martingale measures. A more detailed discussion of some valuation approaches, among them Steffensen (2000) and Becherer (2003), will take place in Section 8.

The present paper works with a discrete finite time framework. Like other papers in this field, it is general in the sense that it does not propose particular models for the dynamics of financial securities or biometric events. The concept of a life insurance policy is introduced in a very general way and the presented methods are not restricted to particular types of contracts. The diversification approach is carried out by assuming certain properties (most of them also assumed in the articles cited above) of the underlying stochastic model, like e.g. independence of individuals, independence of biometric and financial events, no-arbitrage pricing, etc. To be able to model a wide variety of possible types of policies and lives, we assume an infinite product space for the biometric risks that also provides for each possible life (of which we may have infinitely many) infinitely many i.i.d. ones (= large cohorts of similar lives). In fact, the setting is that we consider biometric probability spaces (= lives) and random variables on their products with the financial probability space (= policies). As already said, the resulting product measure valuation principle is in accordance with existing results. Because of no-arbitrage pricing, not only prices at time 0, but complete price processes are determined. Under the mentioned assumptions, it is then shown how a life insurance company can accomplish the earlier described convergence of mean balances of hedges together with contractual payments. The initial costs of the respective purely financial and self-financing hedging strategies can be financed by the minimum fair premiums.

The hedging method considered in this paper is different from the risk-minimizing and mean-variance hedging strategies in Møller (1998, 2001, 2002). In fact, the method is a discrete generalization of the matching approach in Aase and Persson (1994). This method is less sophisticated than e.g. risk minimizing strategies (which are unfortunately not self-financing), but is practicable in the sense that not every single life has to be observed over the whole term. The paper provides examples for pricing and hedging of different types of policies. A more detailed example shows for a term assurance and an endowment the historical development of the ratio of the minimum fair annual premium per benefit. Assuming that premiums are calculated by a conservatively chosen constant technical rate of interest, the example also derives the development of the market values, i.e. minimum fair prices, of these contracts.

The section content is as follows. In Section 2, some principles considered to be reasonable for a basic theory of life insurance are briefly discussed in an enumerated list. Section 3 introduces the market model and the first mathematical assumptions concerning the stochastic model of financial and biometric risks (product space). Section 4 defines general life insurance policies and states a generalized Principle of Equivalence (cf. Persson, 1998). In Section 5, the case of classical life insurance mathematics and the motivation of the Expectation Principle by risk diversification, i.e. the Law of Large Numbers, is briefly reviewed. Section 6 contains the Law of Large Numbers approach to valuation in the general case and the deduction of the minimum fair price (product measure principle). In particular, it is explained how the Strong Law of Large Numbers can be properly applied in the introduced product space framework. Section 7 is about hedging, i.e. about the convergence of mean balances. In this section, examples are given, too. In Section 8, we discuss related results in the present literature on derivation of valuation principles. In Section 9, it is shown how parts of the results can be adapted to the case of incomplete markets. Even for markets with arbitrage opportunities some results still hold. Section 10 is dedicated to the numerical pricing example mentioned above. The last section is the conclusion. The appendix contains figures.

2 Some principles

The following eight principles informally describe the biometric and financial framework of this article. The formulation by mathematical assumptions follows later. It is clear that the principles of our model are not perfect or complete in any sense, and a considerable amount of research is carried out in areas where this might be particularly true, e.g. thinking of the idealized assumption of independent individual biometry (principle 3) which is strongly challenged by the evidence of so-called longevity risk (e.g. Richards and Jones, 2004). However, the proposed model should be seen as a rudimentary life insurance framework inheriting some basic ideas and idealized assumptions from the classical theory, but already working with stochastic financial markets. In this sense, it is a modern framework. Each principle is given with a short explanation motivating it.

1. Independence of biometric and financial events. Biometric (or technical) events, for instance death or injury of persons, are assumed to be stochastically independent of the events of the financial markets (cf. Aase and Persson, 1994). In contrast to reinsurance companies, where the movements on the financial markets can be highly correlated to technical events (e.g. earthquakes), such effects are rather unlikely in the case of life insurance.

2. Complete arbitrage-free financial markets. Except for Section 9, where incomplete markets are examined, complete and arbitrage-free *financial* markets are considered throughout the paper. Even though this might be an unrealistic assumption from the viewpoint of finance, it is realistic from the perspective of life insurance. The reason is that a life insurance company usually does not invent purely financial products as this is the working field of banks. Therefore, it can be assumed that all considered *financial* products are either traded on the market, can be bought from banks or can be replicated by self-financing strategies. Nonetheless, it is self-evident that a claim which also depends on a biometric event (e.g. the death of a person) *can not* be hedged by financial securities, i.e. the *joint* market of financial and biometric risks is *not* complete. In the literature, completeness of financial markets is often assumed by the use of the Black-Scholes model (cf. Aase and Persson (1994), Møller (1998)). However, parts of our results are also valid in the case of incomplete financial markets - which allows for more models. In this case, financial portfolios will be restricted to replicable ones, and also the considered life insurance policies are restricted in a similar way.

3. Biometric states of individuals are independent. This is the standard assumption of classical life insurance. Neglecting the possibility of epidemic diseases or wars, the principle could be held for appropriate in a modern framework, too. However, recently research is carried out on modelling and managing risk caused by major demographic developments (see also the introductory remark above). In this case, more realistic models with dependencies fairly enough should replace the biometric independence assumption. Nonetheless, we stay with this assumption as our main argument uses diversifyability of biometric risks and claims to be traditional with respect to this.

4. Large classes of similar individuals. Applying the Law of Large Numbers in classical life insurance mathematics, an implicit assumption is a large number of persons under contract in a particular company. Even stronger, it can usually be assumed that classes of "similar" persons, e.g. of the same age, gender and health status, are large. An insurance company should be able to cope with such a large cohort of similar persons even if all members of the cohort have the same kind of policy (see also Principle 7 below).

5. Similar individuals can not be distinguished. For fairness reasons, any two individuals with similar biometric development to be expected should pay the same price for the same kind of contract. Furthermore, any activity (e.g. hedging) taken by an insurance company for two individuals holding the same kind of policy is assumed to be identical as long as their possible future biometric development is independently identical from the stochastic point of view.

6. No-arbitrage pricing. As we know from the theory of financial markets, an important property of a reasonable pricing system is the absence of arbitrage, i.e. the absence of riskless wins. In our case, it should not be possible to beat the market by selling and buying life insurance products in e.g. an existing or hypothetical reinsurance market (cf. Delbaen and Haezendonck (1989) and Sondermann (1991)). Hence, any product and cash flow will be priced or given a value under the no-arbitrage principle.

7. Minimum fair prices allow hedging such that mean balances converge to zero almost surely. The principle of independence of the biometric probability spaces is closely related to the Expectation Principle of classical life insurance mathematics. In the classical case, where financial markets are assumed to be deterministic, this principle states that the value or single net premium of a cash flow is the expectation of the sum of its discounted payoffs (*expected present value*). The connection between the two principles is the Law of Large Numbers. Values or prices are determined such that for an increasing number of contracts issued to independent individuals the insurer can accomplish that the mean final balance per policy converges to zero almost surely (the variance of this mean balance converges to null, too). In analogy to the classical case, we generally demand that the minimum fair price of any policy (from the viewpoint of the insurer) should at least cover the price of a purely financial hedging strategy that lets the mean balance per policy converge to zero a.s. for an increasing number of policyholders.

8. Principle of Equivalence. Under a reasonable valuation principle (cf. Principle 7), the Principle of Equivalence demands that the future payments to the insurer (premiums) should be determined such that their (market) value equals the (market) value of the future payments to the insured (benefits). The idea is that the liabilities (benefits, claims) can somehow be hedged working with the premiums. In the coming sections, this concept will be considered in detail.

Remark 1. In the theory of deterministic financial markets, today's (time 0) price of a future cash flow is called its *present value*. The value of a future cash flow also subject to mortality risk, evaluated with the classical Expectation Principle, is called its *expected present value*. The value of the same cash flow evaluated with principles of modern life insurance mathematics (stochastic financial markets) is called *market value* in the literature since evaluation is usually done using market prices of related securities which do not contain biometric risks. However, the notion market value is somehow misleading as the life insurance contracts themselves are usually not traded and hence there usually exist no prices for them that are directly determined by the market. In accordance with Principle 7, we will call the *market value* also the *minimum fair price*.

Remark 2. Concerning premium calculation, the classical Expectation Principle (cf. Principle 7) is usually seen as a minimum premium principle since any insurance company must be able to cope with higher expenses than the expected (cf. Embrechts, 2000). So-called *safety loads* on the minimum fair premiums can be obtained by more elaborate *premium principles*. We refer to the literature for more information on the topic (e.g. Delbaen and Haezendonck (1989); Gerber (1997); Goovaerts, De Vylder and Haezendonck (1984); Møller (2002-2003b); Schweizer (2001)). Another possibility to obtain safety loads is to use the Expectation Principle with a prudent *first order base* (also: technical base or premium basis) for biometric and financial developments, e.g. conservatively chosen mortality and interest rates, that represent a worst-case scenario for the future development of the *second order base* (experience base), that stands for the real, i.e. observed, development (e.g. Norberg, 2001).

3 The model

Let $(F, \mathcal{F}_T, \mathbb{F})$ be a probability space equipped with the filtration $(\mathcal{F}_t)_{t\in\mathbb{T}}$, where $\mathbb{T} = \{0, 1, 2, \ldots, T\}$ denotes the discrete finite time axis. Assume that $\mathcal{F}_0$ is trivial, i.e. $\mathcal{F}_0 = \{\emptyset, F\}$. Let the price dynamics of d securities of a frictionless financial market be given by an adapted $\mathbb{R}^d$-valued process $S = (S_t)_{t\in\mathbb{T}}$. The d assets with price processes $(S_t^0)_{t\in\mathbb{T}}, \ldots, (S_t^{d-1})_{t\in\mathbb{T}}$ are traded at times $t \in \mathbb{T} \setminus \{0\}$. The first asset with price process $(S_t^0)_{t\in\mathbb{T}}$ is called the *money account* and has the properties $S_0^0 = 1$ and $S_t^0 > 0$ for $t \in \mathbb{T}$. The tuple $M^F = (F, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{F}, \mathbb{T}, S)$ is called a *securities market model*. A *portfolio* in M^F is given by a d-dimensional vector $\theta = (\theta^0, \ldots, \theta^{d-1})$ of real-valued random variables θ^i $(i = 0, \ldots, d-1)$ on $(F, \mathcal{F}_T, \mathbb{F})$. A *t-portfolio* is a portfolio θ_t which is $\mathcal{F}_t$-measurable. As usual, $\mathcal{F}_t$ is interpreted as the information available at time t. Since an economic agent takes decisions with respect to the information available, a *trading strategy* is a vector $\theta_\mathbb{T} = (\theta_t)_{t\in\mathbb{T}}$ of t-portfolios θ_t. The discounted total gain (or loss) of such a strategy is given by $\sum_{t=0}^{T-1} \langle \theta_t, \overline{S}_{t+1} - \overline{S}_t \rangle$, where $\overline{S} := (S_t/S_t^0)_{t\in\mathbb{T}}$ denotes the price process discounted by the money account and $\langle ., . \rangle$ denotes the inner product on $\mathbb{R}^d$. One can now define

$$G = \left\{ \sum_{t=0}^{T-1} \langle \theta_t, \overline{S}_{t+1} - \overline{S}_t \rangle : \text{ each } \theta_t \text{ is a } t\text{-portfolio} \right\}. \qquad (1)$$

G is a subspace of the space of all real-valued random variables $L^0(F, \mathcal{F}_T, \mathbb{F})$ where two elements are identified if they are equal $\mathbb{F}$-a.s. The process S satisfies the so-called *no-arbitrage condition* (NA) if $G \cap L^0_+ = \{0\}$, where L^0_+ are the non-negative elements of $L^0(F, \mathcal{F}_T, \mathbb{F})$ (cf. Delbaen, 1999). The Fundamental Theorem of Asset Pricing (Dalang, Morton and Willinger, 1990) states that the price process S satisfies (NA) if and only if there is a probability measure $\mathbb{Q}$ equivalent to $\mathbb{F}$ such that under $\mathbb{Q}$ the process $\overline{S}$ is a martingale. $\mathbb{Q}$ is called *equivalent martingale measure* (EMM), then. Moreover, $\mathbb{Q}$ can be found with bounded Radon-Nikodym derivative $d\mathbb{Q}/d\mathbb{F}$.

DEFINITION 1. *A* **valuation principle** π^F *on a set* Θ *of portfolios in* M^F *is a linear mapping which maps each* $\theta \in \Theta$ *to an adapted* $\mathbb{R}$*-valued stochastic process (= price process)* $\pi^F(\theta) = (\pi^F_t(\theta))_{t \in \mathbb{T}}$ *such that*

$$\pi^F_t(\theta) = \langle \theta, S_t \rangle = \sum_{i=0}^{d-1} \theta^i S^i_t \tag{2}$$

for any $t \in \mathbb{T}$ *for which* θ *is* $\mathcal{F}_t$*-measurable.*

For the moment, the set Θ is not specified any further.

Remark 3. Observe that θ is not indexed with some t as we just assume it to be $\mathcal{F}_T$-measurable in general. For instance, in a case where θ is $\mathcal{F}_T$-measurable, but not $\mathcal{F}_{T-1}$-measurable, the valuation principle π^F would assign a value $\pi^F_{T-1}(\theta)$ to θ although it could not be observed at time $T-1$. This is comparable to the case where we assign a value (at time $T-1$) to an option or insurance contract maturing at time T although we not yet know the final outcome of the contract.

Consider an arbitrage-free market with price process S as given above and a portfolio θ with price process $\pi^F(\theta)$. From the Fundamental Theorem it is known that the enlarged market with price dynamics $S' = ((S^0_t, \ldots, S^{d-1}_t, \pi^F_t(\theta)))_{t \in \mathbb{T}}$ is arbitrage-free if and only if there exists an EMM $\mathbb{Q}$ for $\overline{S'}$, i.e. $\mathbb{Q} \sim \mathbb{F}$ and $\overline{S'}$ a $\mathbb{Q}$-martingale. Hence, one has

$$\pi^F_t(\theta) = S^0_t \cdot \mathbf{E}_{\mathbb{Q}}[\langle \theta, S_T \rangle / S^0_T | \mathcal{F}_t]. \tag{3}$$

It is well-known that the no-arbitrage condition does not imply a unique price process for θ when the portfolio can not be replicated by a self-financing strategy $\theta_{\mathbb{T}}$, i.e. a strategy such that $\langle \theta_{t-1}, S_t \rangle = \langle \theta_t, S_t \rangle$ for each $t > 0$ and $\theta_T = \theta$. However, in a *complete* market M^F, i.e. a market which features a self-financing replicating strategy for *any* portfolio θ (cf. Lemma 1), the no-arbitrage condition implies unique prices (where prices are identified when equal a.s.) and therefore a unique EMM $\mathbb{Q}$. Actually, an arbitrage-free securities market model as introduced above is complete if and only if the set of equivalent martingale measures is a singleton (cf. Harrison and Kreps (1979); Taqqu and Willinger (1987); Dalang, Morton and Willinger (1990)).

We will now introduce assumptions which concern the properties of market models (not of valuation principles) that include biometric events (cf. Principles 1 to 4 of Section 2).

Assume to be given a filtered probability space $(B, (\mathcal{B}_t)_{t\in\mathbb{T}}, \mathbb{B})$ which describes the development of the biological states of all considered human beings. *No particular model for the development of the biometric information is assumed.*

ASSUMPTION 1. *A common filtered probability space*

$$(M, (\mathcal{M}_t)_{t\in\mathbb{T}}, \mathbb{P}) = (F, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{F}) \otimes (B, (\mathcal{B}_t)_{t\in\mathbb{T}}, \mathbb{B}) \tag{4}$$

of financial and biometric events is given, i.e. $M = F \times B$, $\mathcal{M}_t = \mathcal{F}_t \otimes \mathcal{B}_t$ *and* $\mathbb{P} = \mathbb{F} \otimes \mathbb{B}$. *Furthermore,* $\mathcal{F}_0 = \{\emptyset, F\}$ *and* $\mathcal{B}_0 = \{\emptyset, B\}$.

As $\mathcal{M}_0 = \{\emptyset, F \times B\}$, the model implies that the world is known for sure at time 0. The symbols $M, \mathcal{M}_t$ and $\mathbb{P}$ are introduced to shorten notation. M and $\mathcal{M}_t$ are chosen since these objects describe events of the underlying market *m*odel, whereas $\mathbb{P}$ denotes the *p*hysical probability measure. Later, $\mathbb{M}$ is used to denote a *m*artingale measure.

ASSUMPTION 2. *A complete securities market model*

$$M^F = (F, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{F}, \mathbb{T}, {}_FS) \tag{5}$$

with a unique equivalent martingale measure $\mathbb{Q}$ *is given. The common market model for financial and biometric risks is denoted by*

$$M^{F\times B} = (M, (\mathcal{M}_t)_{t\in\mathbb{T}}, \mathbb{P}, \mathbb{T}, S), \tag{6}$$

where $S(f, b) = {}_FS(f)$ *for all* $(f, b) \in M$.

In the following, $M^{F\times B}$ is understood as a securities market model. The notions portfolio, no-arbitrage etc. are used as introduced at the beginning of this section. We will need the following lemma.

LEMMA 1.

(*i*) *Any* $\mathcal{F}_t$*-measurable portfolio can be replicated by a self-financing strategy in* M^F *until* t.

(*ii*) *Any* $\mathcal{F}_t$*-measurable payoff can be replicated by a self-financing strategy in* M^F *until* t.

Proof. (i) As M^F is complete, any $\mathcal{F}_T$-measurable payoff X at T can be replicated until T. This is the usual definition of the completeness of a securities market model. Hence, there exists for any $\mathcal{F}_t$-measurable portfolio θ_t a replicating self-financing (s.f.) strategy $(\varphi_t)_{t\in\mathbb{T}}$ in M^F, i.e. $\varphi_T = \theta_t$, since $X = \langle\theta_t, S_T\rangle$ could be chosen. For no-arbitrage reasons, one must have $\pi^F_s(\theta_t) = \langle\varphi_s, S_s\rangle$ for $s \in \mathbb{T}$ and therefore $\langle\theta_t, S_s\rangle = \langle\varphi_s, S_s\rangle$ for any $s \geq t$. So, there also exists a s.f. strategy such that $\varphi_t = \theta_t$, i.e. the portfolio θ_t is replicated until t.
(ii) Due to (i), the portfolio $\theta_t = X/S^0_t \cdot e_0$ can for any $\mathcal{F}_t$-measurable payoff X be replicated until t. Observe that $\langle\theta_t, S_t\rangle = X$. □

Remark 4. S is the canonical embedding of ${}_FS$ into $(M, (\mathcal{M}_t)_{t\in\mathbb{T}}, \mathbb{P})$. We will usually use the same symbol for a random variable X in $(F, \mathcal{F}_t, \mathbb{F})$ and a random variable Y in $(M, \mathcal{M}_t, \mathbb{P})$ $(t \in \mathbb{T})$ when Y is the embedding of X into $(M, \mathcal{M}_t, \mathbb{P})$, i.e. $Y(f, b) = X(f)$ for all $(f, b) \in M$. Now, any portfolio ${}_F\theta$ of the complete financial market M^F can be replicated by some self-financing trading strategy ${}_F\theta_{\mathbb{T}} = ({}_F\theta_t)_{t\in\mathbb{T}}$. Under (NA), the unique price process $\pi^F({}_F\theta)$ of the portfolio is given by

$$\pi_t^F({}_F\theta) = {}_FS_t^0 \cdot \mathbf{E}_{\mathbb{Q}}[\langle {}_F\theta, {}_FS_T\rangle / {}_FS_T^0 | \mathcal{F}_t]. \tag{7}$$

Since S is the embedding of ${}_FS$ into $(M, (\mathcal{M}_t)_{t\in\mathbb{T}}, \mathbb{P})$, the embedded portfolio ${}_F\theta$ in $M^{F\times B}$ is replicated by the embedded trading strategy ${}_F\theta_{\mathbb{T}} = ({}_F\theta_t)_{t\in\mathbb{T}}$ in $M^{F\times B}$. Hence, to avoid arbitrage opportunities, any reasonable valuation principle π must feature a price process $\pi({}_F\theta)$ in $M^{F\times B}$ that fulfills $\pi_t({}_F\theta) = \pi_t^F({}_F\theta)$ $\mathbb{P}$-a.s. for any $t \in \mathbb{T}$. Since $\mathbf{E}_{\mathbb{Q}}[X|\mathcal{F}_t] = \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[X|\mathcal{F}_t \otimes \mathcal{B}_0]$ $\mathbb{P}$-a.s. for *any* random variable X in $(F, \mathcal{F}_T, \mathbb{F})$, one must have $\mathbb{P}$-a.s

$$\begin{aligned} \pi_t({}_F\theta) &= S_t^0 \cdot \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\langle {}_F\theta, S_T\rangle / S_T^0 | \mathcal{F}_t \otimes \mathcal{B}_0] \\ &= S_t^0 \cdot \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\langle {}_F\theta, S_T\rangle / S_T^0 | \mathcal{F}_t \otimes \mathcal{B}_t]. \end{aligned} \tag{8}$$

Observe that $(S_t/S_t^0)_{t\in\mathbb{T}}$ is a $\mathbb{Q}\otimes\mathbb{B}$-martingale.

ASSUMPTION 3. *There are infinitely many human individuals and we have*

$$(B, (\mathcal{B}_t)_{t\in\mathbb{T}}, \mathbb{B}) = \bigotimes_{i=1}^{\infty} (B^i, (\mathcal{B}_t^i)_{t\in\mathbb{T}}, \mathbb{B}^i), \tag{9}$$

where $B_H = \{(B^i, (\mathcal{B}_t^i)_{t\in\mathbb{T}}, \mathbb{B}^i) : i \in \mathbb{N}^+\}$ *is the set of filtered probability spaces describing the development of the i-th individual* $(\mathbb{N}^+ := \mathbb{N} \setminus \{0\})$. *Each* $\mathcal{B}_0^i$ *is trivial.*

It follows that $\mathcal{B}_0$ is also trivial, i.e. $\mathcal{B}_0 = \{\emptyset, B\}$.

ASSUMPTION 4. *For any space* $(B^i, (\mathcal{B}_t^i)_{t\in\mathbb{T}}, \mathbb{B}^i)$ *in* B_H *there are infinitely many isomorphic (=identical, except for the indices) ones in* B_H.

In the sense of Remark 2, the four assumptions above define a model for the second order base.

4 Life insurance policies

Under the setup given in the last section, the biometric development has by definition no influence on the price process S of the financial market - and vice versa. We therefore have situations where a portfolio θ that contains biometric risk - that is a portfolio which is not of the form $\theta = {}_F\theta$ $\mathbb{P}$-a.s. with ${}_F\theta$ an M^F-portfolio - can not be replicated by purely financial products. Hence, in general, relative pricing of life insurance products with respect to M^F is not possible. Usually, life insurance policies are not traded and the possibility of the valuation of such contracts by the market is not given. The market $M^{F\times B}$ of financial and biometric risks is incomplete. Nonetheless, products have to be priced as e.g. the insured usually have the right to dissolve any contract at any time of its duration. We are therefore in the need of a reasonable valuation principle π for the considered portfolios Θ of the market $M^{F\times B}$ and in particular for general life insurance products.

DEFINITION 2. *A general* **life insurance policy** *is a vector* $(\gamma_t, \delta_t)_{t\in\mathbb{T}}$ *of pairs* (γ_t, δ_t) *of* t*-portfolios in* Θ *(to shorten notation we drop the inner brackets of* $((\gamma_t, \delta_t))_{t\in\mathbb{T}}$*). For any* $t \in \mathbb{T}$*, the portfolio* γ_t *is interpreted as a payment of the insurer to the insurant* (**benefit**) *and* δ_t *as a payment of the insurant to the insurer* (**premium**)*, respectively taking place at* t*. The notation* $({}^i\gamma_t, {}^i\delta_t)_{t\in\mathbb{T}}$ *means that the contract depends on the* i*-th individual's life, i.e. for all* $(f, x), (f, y) \in M$

$$({}^i\gamma_t(f, x), {}^i\delta_t(f, x))_{t\in\mathbb{T}} = ({}^i\gamma_t(f, y), {}^i\delta_t(f, y))_{t\in\mathbb{T}} \tag{10}$$

whenever $p^i(x) = p^i(y)$*,* p^i *being the canonical projection of* B *onto* B^i*.*

For any policy $(\gamma_t, \delta_t)_{t\in\mathbb{T}}$ issued by a life office to an individual, this stream of payments is from the viewpoint of the insurer equivalent to holding the portfolios $(\delta_t - \gamma_t)_{t\in\mathbb{T}}$.

Although there has not yet been considered any particular valuation principle, it is assumed that a suitable principle π is a minimum fair price in the heuristic sense given in Section 2, Principle 7. The properties of a minimum fair price will be defined and further explained in Section 6.

ASSUMPTION 5. *Suppose a suitable valuation principle* π *on* Θ*. For any life insurance policy* $(\gamma_t, \delta_t)_{t\in\mathbb{T}}$ *the* **Principle of Equivalence** *demands that*

$$\pi_0\left(\sum_{t=0}^{T} \gamma_t\right) = \pi_0\left(\sum_{t=0}^{T} \delta_t\right). \tag{11}$$

As already mentioned in Section 2 (Principle 8), the idea of Eq. (11) is that the liabilities $(\gamma_t)_{t\in\mathbb{T}}$ can somehow be hedged working with the premiums $(\delta_t)_{t\in\mathbb{T}}$ since their present values or market values are identical. For the classical case, this idea is explained in the next section.

Remark 5. We use portfolio notation (and not cash flow notation) since e.g. a unit-linked life insurance policy depends usually on shares of a fund which are combinations of traded assets. Trading and hedging strategies become more transparent with this notation. Furthermore, later stated integrability conditions can be formulated for units of a portfolio, rather than for a combined general cash flow. We think, this makes the application of these conditions easier (compare Examples 3, 4, and Remark 10).

Since portfolio notation is not commonly used in life insurance mathematics, we give a brief example of an application of Eq. (11) for a unit-linked assurance.

Example 1 (Portfolio notation for a unit-linked assurance). We use Assumptions 1 and 2. For $T = 2$ and $d = 2$, assume a complete and arbitrage-free securities market model M^F, e.g. a Cox-Ross-Rubinstein model, with two assets, the first one being deterministic (bond, S^0), the second one stochastic (stock, S^1). We consider a life aged x. This life is modelled by the filtered space $(B, (\mathcal{B}_t)_{t\in\{0,1,2\}}, \mathbb{B})$, and we assume that he or she is alive at time 0. Suppose now that β is $\mathcal{B}_1$-measurable with $\beta(b) \in \{0, 1\}$ for any $b \in B$. Assume that $\beta = 1$ if and only if the individual is alive at 1. Define now $\gamma_0 = (0, 0)$, $\gamma_1 = (0, 1000(1 - \beta))$ and $\gamma_2 = (0, 1500\beta)$. Furthermore, $\delta_0 = (P, 0)$, where $P \in \mathbb{R}$, and $\delta_1 = \delta_2 = (0, 0)$. These portfolios define a simple unit-linked assurance for the considered life. The policy features a single premium of P at time 0, the benefit of 1000 shares at time 1 if the life has died until then, or, if this was not the case, the benefit of 1500 shares at time 2. For the calculation of P by (11), we will use the product measure principle, which will be discussed in detail later. Hence, we assume that π is given by

$$\pi_t(\theta) = S_t^0 \cdot \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\langle\theta, S_2\rangle/S_2^0|\mathcal{F}_t \otimes \mathcal{B}_t], \quad t \in \{0, 1, 2\}. \tag{12}$$

Therefore,

$$\begin{aligned} \pi_0\left(\sum_{t=0}^{2}\gamma_t\right) &= \pi_0\left(\sum_{t=0}^{2}\delta_t\right) \\ \pi_0\left(1000(0, 1 - \beta + 1.5\beta)\right) &= \pi_0\left((P, 0)\right) \\ 1000\mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[(1 + 0.5\beta)S_2^1/S_2^0] &= P. \end{aligned} \tag{13}$$

From this we obtain

$$\begin{aligned} P &= 1000\mathbf{E}_{\mathbb{B}}[1 + 0.5\beta]\mathbf{E}_{\mathbb{Q}}[S_2^1/S_2^0] \\ &= 1000(1 + 0.5\mathbf{E}_{\mathbb{B}}[\beta])S_0^1 \\ &= 1000S_0^1 + p_x 500S_0^1, \end{aligned} \tag{14}$$

where $p_x = \mathbf{E}_{\mathbb{B}}[\beta]$ is international actuarial notation for the probability that an individual aged x survives the following year. Hence, the single premium is 1000 times the share price at time zero, plus 500 times the share price at time 0 multiplied with the one-year survival probability of the life (x). This reflects the policy, that pays 1000 shares for sure, either at time 1 or at time 2, and an additional 500 shares at time 2 if the policyholder survived the first year.

5 Valuation: the classical case

In classical life insurance mathematics, the financial market is assumed to be deterministic. We realize this assumption by $|\mathcal{F}_T| = 2$, i.e. $\mathcal{F}_T = \{\emptyset, F\}$, and identify $(M, (\mathcal{M}_t)_{t\in\mathbb{T}}, \mathbb{P})$ with $(B, (\mathcal{B}_t)_{t\in\mathbb{T}}, \mathbb{B})$. As the market is assumed to be free of arbitrage, all assets must have the same dynamics up to scaling factors. Hence, we can assume $S = (S_t^0)_{t\in\mathbb{T}}$, i.e. $d = 1$ and the only asset is the money account as a deterministic function of time. In the classical framework, it is common sense that the fair value (or price) at time s of a $\mathbb{B}$-integrable payoff C_t at t is the conditional expectation of the discounted payoff with respect to $\mathcal{B}_s$, i.e. for a t-portfolio C_t/S_t^0, having the value C_t at t, we have

$$\pi_s(C_t/S_t^0) := S_s^0 \cdot \mathbf{E}_{\mathbb{B}}[C_t/S_t^0|\mathcal{B}_s], \quad s \in \mathbb{T}. \tag{15}$$

Under the *Expectation Principle* (15), the classical Principle of Equivalence is given by (11). As the discounted price processes are $\mathbb{B}$-martingales, the classical financial market together with a finite number of classical price processes of life policies is free of arbitrage opportunities.

Let us have a closer look at the logic of valuation principle (15). Assume that Θ is given by the $\mathbb{B}$-integrable portfolios. Suppose Assumption 1 to 3 and consider the claims $\{(-{}^i\gamma_t)_{t\in\mathbb{T}} : i \in \mathbb{N}^+\}$ of a policy from the companies point of view, where ${}^i\gamma_t$ depends on the i-th individual's life, only (cf. Definition 2). Furthermore, suppose that for all $t \in \mathbb{T}$ there is a $c_t \in \mathbb{R}^+$ such that

$$||{}^i\gamma_t||_2 \leq c_t \tag{16}$$

for all $i \in \mathbb{N}^+$, where $||.||_2$ denotes the L^2-norm of the Hilbert space $L^2(M, \mathcal{M}_T, \mathbb{P})$ of all square-integrable real functions on $(M, \mathcal{M}_T, \mathbb{P})$. Now, buy for all $i \in \mathbb{N}^+$ and all $t \in \mathbb{T}$ the portfolios $\mathbf{E}_{\mathbb{B}}[{}^i\gamma_t]$, where $\mathbf{E}_{\mathbb{B}}[{}^i\gamma_t]$ is interpreted as a financial product (a t-portfolio) which matures at time t, i.e. the payoff $\mathbf{E}_{\mathbb{B}}[{}^i\gamma_t] \cdot S_t^0$ in cash at t is bought at 0. Consider the balance of wins and losses at time t. The mean total payoff at t for the first m policies is given by

$$\frac{1}{m}\sum_{i=1}^{m}(\mathbf{E}_{\mathbb{B}}[{}^i\gamma_t] - {}^i\gamma_t) \cdot S_t^0. \tag{17}$$

Clearly, (17) converges $\mathbb{B}$-a.s. to 0 as we can apply the SLLN by Kolmogorov's Criterion (cf. (16)). Furthermore, it follows directly from (15) that we have $\pi_0(\mathbf{E}_{\mathbb{B}}[{}^i\gamma_t]) = \pi_0({}^i\gamma_t)$ for all $i \in \mathbb{N}^+$. Hence, in the classical case, the fair value of any claim equals (except for the different sign, perhaps) the price of a hedge at time 0 such that for an increasing number of independent claims the mean balance of claims and hedges converges to zero almost surely.

Now, consider the set of life insurance contracts $\{({}^i\gamma_t, {}^i\delta_t)_{t\in\mathbb{T}} : i \in \mathbb{N}^+\}$ with the deltas being defined in analogy to the gammas above. Since for the company a policy can be considered as a vector $({}^i\delta_t - {}^i\gamma_t)_{t\in\mathbb{T}}$ of portfolios, the analogous hedge is given by $(\mathbf{E}_{\mathbb{B}}[{}^i\gamma_t] - \mathbf{E}_{\mathbb{B}}[{}^i\delta_t])_{t\in\mathbb{T}}$. Under Assumption 5 the policy has value zero. From the Expectation Principle (15) we therefore obtain for all $i \in \mathbb{N}^+$

$$\sum_{t=0}^{T}\pi_0(\mathbf{E}_{\mathbb{B}}[{}^i\delta_t] - \mathbf{E}_{\mathbb{B}}[{}^i\gamma_t]) = \sum_{t=0}^{T}\pi_0({}^i\delta_t - {}^i\gamma_t) = 0. \tag{18}$$

Hence, under (15) and Assumptions 1, 2, 3 and 5, a life office can (without any costs at time 0) pursue a hedge such that the mean balance per contract at any time t converges to zero almost surely for an increasing number of individual policies:

$$\frac{1}{m}\sum_{i=1}^{m}({}^{i}\delta_t - {}^{i}\gamma_t - \mathbf{E}_{\mathbb{B}}[{}^{i}\delta_t] + \mathbf{E}_{\mathbb{B}}[{}^{i}\gamma_t]) \cdot S_t^0 \stackrel{m\to\infty}{\longrightarrow} 0 \quad \mathbb{B}\text{-a.s.} \tag{19}$$

As a direct consequence, the mean of the *final* balance converges, too:

$$\frac{1}{m}\sum_{i=1}^{m}\sum_{t=0}^{T}({}^{i}\delta_t - {}^{i}\gamma_t - \mathbf{E}_{\mathbb{B}}[{}^{i}\delta_t] + \mathbf{E}_{\mathbb{B}}[{}^{i}\gamma_t]) \cdot S_T^0 \stackrel{m\to\infty}{\longrightarrow} 0 \quad \mathbb{B}\text{-a.s.} \tag{20}$$

Remark 6. Roughly speaking, the Expectation Principle (15) implies that the price of any claim at least covers the costs of a purely financial hedge such that for an increasing number of independent claims the mean balance of claims and hedges converges to zero almost surely. This is how *diversification* of biometric risks appears in the classical case. Under the Equivalence Principle (11), the hedge of any insurance contract costs nothing at time 0, which is important as the contract itself is for free, too (cf. Eq. (18)).

6 Valuation: the general case

Before it comes to the topic of valuation in the general case, two technical lemmas have to be proven and some further notation has to be introduced.

Let the set $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ be equipped with the usual Borel-σ-algebra and recall that a function g into $\overline{\mathbb{R}}$ is called *numeric*.

LEMMA 2. *Consider $n > 1$ measurable numeric functions g_1 to g_n on the product $(F, \mathcal{F}, \mathbb{F}) \otimes (B, \mathcal{B}, \mathbb{B})$ of two arbitrary probability spaces. Then $g_1 = \ldots = g_n$ $\mathbb{F} \otimes \mathbb{B}$-a.s. if and only if $\mathbb{F}$-a.s. $g_1(f, .) = \ldots = g_n(f, .)$ $\mathbb{B}$-a.s.*

Proof. For any $Q \in \mathcal{F} \otimes \mathcal{B}$ it is well-known that $\mathbb{F} \otimes \mathbb{B}(Q) = \int \mathbb{B}(Q_f)d\mathbb{F}$, where $Q_f = \{b \in B : (f, b) \in Q\}$ and the function $\mathbb{B}(Q_f)$ on F is $\mathcal{F}$-measurable. As for $i \neq j$ the difference $g_{i,j} := g_i - g_j$ is measurable, the set $Q := \bigcap_{i \neq j} g_{i,j}^{-1}(0)$ is $\mathcal{F} \otimes \mathcal{B}$-measurable. Now, $g_1 = \ldots = g_n$ a.s. is equivalent to $\mathbb{F} \otimes \mathbb{B}(Q) = 1$ and this again is equivalent to $\mathbb{B}(Q_f) = 1$ $\mathbb{F}$-a.s. However, $\mathbb{B}(Q_f) = 1$ is equivalent to $g_1(f, .) = \ldots = g_n(f, .)$ $\mathbb{B}$-a.s. □

LEMMA 3. *Let $(g_n)_{n \in \mathbb{N}}$ and g be a sequence, respectively a function, in $L^0(F \times B, \mathcal{F} \otimes \mathcal{B}, \mathbb{F} \otimes \mathbb{B})$, i.e. the real valued measurable functions on $F \times B$, where $(F \times B, \mathcal{F} \otimes \mathcal{B}, \mathbb{F} \otimes \mathbb{B})$ is the product of two arbitrary probability spaces. Then $g_n \to g$ $\mathbb{F} \otimes \mathbb{B}$-a.s. if and only if $\mathbb{F}$-a.s. $g_n(f, .) \to g(f, .)$ $\mathbb{B}$-a.s.*

Proof. The elements of $L^0(F \times B, \mathcal{F} \otimes \mathcal{B}, \mathbb{F} \otimes \mathbb{B})$ are measurable numeric functions. Now, recall that for any sequence of real numbers $(h_n)_{n\in\mathbb{N}}$ and any $h \in \mathbb{R}$ the property $h_n \to h$ is equivalent to $\limsup h_n = \liminf h_n = h$. As the limes superior and the limes inferior of a measurable numeric function always exist and are measurable, one obtains from Lemma 2 that

$$\limsup_{n\to\infty} g_n = \liminf_{n\to\infty} g_n = g \quad \mathbb{F} \otimes \mathbb{B}\text{-a.s.} \tag{21}$$

if and only if $\mathbb{F}$-a.s.

$$\limsup_{n\to\infty} g_n(f,.) = \liminf_{n\to\infty} g_n(f,.) = g(f,.) \quad \mathbb{B}\text{-a.s.} \tag{22}$$

□

As we have seen in Section 4, there is the need for a suitable set Θ of portfolios on which a particular valuation principle will work. Furthermore, a mathematically precise description of what was called “similar” in Principle 5 (Section 2) has to be introduced.

DEFINITION 3.

(*i*) *Define*

$$\Theta = (L^1(M, \mathcal{M}_T, \mathbb{P}))^d \tag{23}$$

and

$$\Theta^F = (L^1(F, \mathcal{F}_T, \mathbb{F}))^d, \tag{24}$$

where Θ^F can be interpreted as a subset of Θ by the usual embedding.

(*ii*) *A set $\Theta' \subset \Theta$ of portfolios in $M^{F\times B}$ is called* independently identically distributed with respect to $(B, \mathcal{B}_T, \mathbb{B})$*, abbreviated* **B-i.i.d.**, *when for almost all $f \in F$ the random variables $\{\theta(f,.) : \theta \in \Theta'\}$ are i.i.d. on $(B, \mathcal{B}_T, \mathbb{B})$. Under Assumption 4, such sets exist and can be countably infinite.*

(*iii*) *Under the Assumptions 1 to 3, a set $\Theta' \subset \Theta$ satisfies condition* **(K)** *if for almost all $f \in F$ the elements of $\{\theta(f,.) : \theta \in \Theta'\}$ are stochastically independent on $(B, \mathcal{B}_T, \mathbb{B})$ and $||\theta^j(f,.)||_2 < c(f) \in \mathbb{R}^+$ for all $\theta \in \Theta'$ and all $j \in \{0, \ldots, d-1\}$.*

Sets fulfilling condition (B-i.i.d.) or (K) are indexed with the respective symbol. A discussion of the Kolmogorov Criterion like condition (K) can be found below (Remark 10). The condition figures out to be quite weak with respect to all relevant practical purposes.

The remaining assumptions concerning valuation can be stated now. The next assumption is motivated by the demand that whenever the market with the original d securities with prices S is enlarged by a finite number of price processes $\pi(\theta)$ due to general portfolios $\theta \in \Theta$, the no-arbitrage condition (NA) should hold for the new market. This assumption corresponds to Principle 6 in Section 2.

ASSUMPTION 6. *Any valuation principle π taken into consideration must for any $t \in \mathbb{T}$ and $\theta \in \Theta$ be of the form*

$$\pi_t(\theta) = S_t^0 \cdot \mathbf{E}_{\mathbb{M}}[\langle \theta, S_T \rangle / S_T^0 | \mathcal{F}_t \otimes \mathcal{B}_t] \tag{25}$$

for a probability measure $\mathbb{M} \sim \mathbb{P}$. Furthermore, one must have

$$\pi_t({}_F\theta) = \pi_t^F({}_F\theta) \tag{26}$$

$\mathbb{P}$*-a.s. for any M^F-portfolio ${}_F\theta$ and all $t \in \mathbb{T}$, where π_t^F is as in (7).*

Observe that by Assumption 6, the process $(S_t/S_t^0)_{t\in\mathbb{T}}$ must be an $\mathbb{M}$-martingale. To see that use (25) and (26) with ${}_F\theta = e_{i-1}$ (i-th canonical base vector in $\mathbb{R}^d$) and apply (2).

The following assumption is regarding the fifth and the seventh principle.

ASSUMPTION 7. *Under the Assumptions 1 - 4 and 6, a* **minimum fair price** *is a valuation principle π on Θ that must for any $\theta \in \Theta$ fulfill*

$$\pi_0(\theta) = \pi_0^F(H(\theta)), \tag{27}$$

where

$$H : \Theta \longrightarrow \Theta^F \tag{28}$$

is such that

(i) *$H(\theta)$ is a t-portfolio whenever θ is.*

(ii) *$H({}^1\theta) = H({}^2\theta)$ for B-i.i.d. portfolios ${}^1\theta$ and ${}^2\theta$.*

(iii) *for t-portfolios $\{{}^i\theta : i \in \mathbb{N}^+\}_{B-i.i.d.}$ or $\{{}^i\theta : i \in \mathbb{N}^+\}_K$, one has*

$$\frac{1}{m}\sum_{i=1}^{m} \langle {}^i\theta - H({}^i\theta), S_t \rangle \overset{m\to\infty}{\longrightarrow} 0 \quad \mathbb{P}\text{-}a.s. \tag{29}$$

Relation (28) means that the *hedge* $H(\theta)$ is a portfolio of the *financial* market. Recall that the financial market M^F is complete and any t-portfolio features a self-financing replicating strategy until time t (cf. Lemma 1). However, (28) also implies that the hedging strategy does not react on biometric events happening after time 0. Due to (ii), as in the classical case, the *hedging method* H can not distinguish between similar (B-i.i.d.) individuals (cf. Principle 5). Property (iii) is also adopted from the classical case, where pointwise convergence is ensured by the Expectation Principle for appropriate insurance products combined with respective hedges (cf. Principle 7 and Section 5). Property (iii) is also related to Principle 4 in Section 2 as insurance companies should be able to cope with large classes of similar (B-i.i.d.) contracts.

Now, the main result of this paper can be stated.

THEOREM 1. *Under the Assumptions 1 to 4, 6 and 7, the minimum fair price π on Θ is uniquely determined by $\mathbb{M} = \mathbb{Q} \otimes \mathbb{B}$, i.e. for $\theta \in \Theta$ and $t \in \mathbb{T}$,*

$$\pi_t(\theta) = S_t^0 \cdot \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\langle \theta, S_T \rangle / S_T^0 | \mathcal{F}_t \otimes \mathcal{B}_t]. \tag{30}$$

As already has been mentioned, this valuation principle is quite well established in the literature. However, our mathematically detailed derivation within a very general framework seems to be new (see also Section 1, resp. 8). Clearly, (15) is the special case of (30) in the presence of a deterministic financial market (e.g. when $|\mathcal{F}_T| = 2$). As π is unique, it is at the same time the minimal valuation principle with the demanded properties. There is no other valuation principle under the setting of Assumptions 1 - 4 that fulfills 6 and 7 and implies under the Principle of Equivalence (Assumption 5) lower premiums than (30). Actually, property (iii) of Assumption 7 ensures that insurance companies do not charge more than the cost of a more or less acceptable purely financial hedge for each product which is sold. So to speak, the minimum fair price is fair from the viewpoint of the insured, as well as from the viewpoint of the companies.

The following lemmas are needed in order to prove the theorem.

LEMMA 4. *On $(F \times B, \mathcal{F}_T \otimes \mathcal{B}_T)$, it holds that*

$$\mathbb{Q} \otimes \mathbb{B} \sim \mathbb{F} \otimes \mathbb{B}. \tag{31}$$

For the Radon-Nikodym derivatives, one has $\mathbb{F} \otimes \mathbb{B}$-a.s.

$$\frac{d(\mathbb{Q} \otimes \mathbb{B})}{d(\mathbb{F} \otimes \mathbb{B})} = \frac{d\mathbb{Q}}{d\mathbb{F}}. \tag{32}$$

Proof. For any $\mathcal{F}_T \otimes \mathcal{B}_T$-measurable set Z, one has $\mathbb{Q} \otimes \mathbb{B}(Z) = 0$ if and only if $\mathbf{1}_Z = 0$ $\mathbb{Q} \otimes \mathbb{B}$-a.s. for the indicator function $\mathbf{1}_Z$ of Z. However, $\mathbf{1}_Z = 0$ $\mathbb{Q} \otimes \mathbb{B}$-a.s. if and only if $\mathbb{Q}$-a.s. $\mathbf{1}_Z(f, .) = 0$ $\mathbb{B}$-a.s. due to Lemma 2. But $\mathbb{Q} \sim \mathbb{F}$, i.e. $\mathbb{Q}$-a.s. and $\mathbb{F}$-a.s. are equivalent, and $\mathbb{Q} \otimes \mathbb{B}(Z) = 0$ equivalent to $\mathbb{F} \otimes \mathbb{B}(Z) = 0$ follows. Hence, (31). For any $\mathcal{F}_T \otimes \mathcal{B}_T$-measurable set Z,

$$\mathbb{Q} \otimes \mathbb{B}(Z) = \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\mathbf{1}_Z] = \mathbf{E}_{\mathbb{Q}}[\mathbf{E}_{\mathbb{B}}[\mathbf{1}_Z]] \tag{33}$$

due to Fubini's Theorem. From the Fundamental Theorem $d\mathbb{Q}/d\mathbb{F}$ exists and is bounded, i.e.

$$\mathbb{Q} \otimes \mathbb{B}(Z) = \mathbf{E}_{\mathbb{F}}\left[\frac{d\mathbb{Q}}{d\mathbb{F}} \mathbf{E}_{\mathbb{B}}[\mathbf{1}_Z]\right] = \mathbf{E}_{\mathbb{F}\otimes\mathbb{B}}\left[\mathbf{1}_Z \frac{d\mathbb{Q}}{d\mathbb{F}}\right]. \tag{34}$$

□

LEMMA 5. *Under Assumption 1 and 2, one has for any $\theta \in \Theta$*

$$H^*(\theta) := \mathbf{E}_{\mathbb{B}}[\theta] \in \Theta^F. \tag{35}$$

There is a self-financing strategy replicating $H^*(\theta)$, and under Assumption 6

$$\pi_t(H^*(\theta)) = S_t^0 \cdot \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\langle \theta, S_T \rangle / S_T^0 | \mathcal{F}_t \otimes \mathcal{B}_0] \tag{36}$$

for $t \in \mathbb{T}$. Moreover, H^* fulfills properties (i), (ii) and (iii) of Assumption 7.

Proof. By Fubini's Theorem, $\mathbf{E}_{\mathbb{B}}[\theta(f,.)]$ exists $\mathbb{F}$-a.s. and $\mathbf{E}_{\mathbb{B}}[\theta]$ is $\mathbb{F}$-measurable and -integrable. Hence, by the completeness of M^F and uniqueness of $\mathbb{Q}$, the portfolio (35) can be replicated by the financial securities in M^F and has due to Assumption 6 and Remark 4 the price process

$$\pi_t(\mathbf{E}_{\mathbb{B}}[\theta]) = S_t^0 \cdot \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\langle \mathbf{E}_{\mathbb{B}}[\theta], S_T\rangle / S_T^0 | \mathcal{F}_t \otimes \mathcal{B}_0]. \tag{37}$$

$\langle \theta, S_T\rangle / S_T^0$ is $\mathbb{F}\otimes\mathbb{B}$-integrable, since each θ^i $(i = 0,\ldots,d-1)$ is $\mathbb{F}\otimes\mathbb{B}$-integrable, $S_T^0 > 0$, and S_T^0 almost surely takes finites values only (Dalang, Morton and Willinger (1990)). By Lemma 4, (36) exists as (32) is bounded. Since $\mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\mathbf{E}_{\mathbb{B}}[X]|\mathcal{F}_t\otimes\mathcal{B}_0] = \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[X|\mathcal{F}_t\otimes\mathcal{B}_0]$ $\mathbb{P}$-a.s. for any $\mathbb{Q}\otimes\mathbb{B}$-integrable X (recall that $\mathcal{B}_0 = \{0, B\}$), (37) is identical to (36) $\mathbb{P}$-a.s. As we have $\mathbf{E}_{\mathbb{B}}[X] = \mathbf{E}_{\mathbb{F}\otimes\mathbb{B}}[X|\mathcal{F}_t\otimes\mathcal{B}_0]$ $\mathbb{P}$-a.s. for $\mathcal{F}_t\otimes\mathcal{B}_t$-measurable X, $H^*(\theta)$ is a t-portfolio. Property (ii) of Assumption 7 is obviously fulfilled. For any t-portfolios $\{{}^i\theta : i\in\mathbb{N}^+\}_K$ or $\{{}^i\theta : i\in\mathbb{N}^+\}_{B-i.i.d.}$, the SLLN (in the first case by Kolmogorov's Criterion) implies for almost all $f\in F$ that

$$\frac{1}{m}\sum_{i=1}^{m}\langle {}^i\theta(f,.) - H^*({}^i\theta)(f), S_t(f)\rangle \overset{m\to\infty}{\longrightarrow} 0 \quad \mathbb{B}\text{-a.s.} \tag{38}$$

Lemma 3 completes the proof. □

LEMMA 6. *Under Assumption 1 and 2, for any $\theta\in\Theta$, any $t\in\mathbb{T}$ and for* $\mathbb{M}\in\{\mathbb{F}\otimes\mathbb{B}, \mathbb{Q}\otimes\mathbb{B}\}$

$$\mathbf{E}_{\mathbb{M}}[\langle \theta - H^*(\theta), S_t\rangle] = 0. \tag{39}$$

Proof. By Fubini's Theorem. □

LEMMA 7. *Under the Assumptions 1 - 4 and 6, any $H : \Theta\to\Theta^F$ fulfilling (i), (ii) and (iii) of Assumption 7 fulfills for any θ in some $\Theta_{B-i.i.d.}$*

$$\pi_t(H(\theta)) = S_t^0 \cdot \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\langle \theta, S_T\rangle / S_T^0 | \mathcal{F}_t\otimes\mathcal{B}_0], \quad t\in\mathbb{T}. \tag{40}$$

The lemma shows for portfolios that could represent life policies that any purely financial hedging method (i.e. a strategy not using biometric information) fulfilling (i), (ii) and (iii) of Assumption 7 has the same price process as (35). In particular, there is no such hedging method with stronger convergence properties than (35).

Proof of Lemma 7. Consider to be given such an H as in Lemma 7 and a set $\{{}^i\theta, i\in\mathbb{N}^+\}_{B-i.i.d.}$ of portfolios that contains a given portfolio $\theta\in\Theta$. As any $\theta\in\Theta$ is a T-portfolio, Lemma 3 implies that $\mathbb{F}$-a.s.

$$\frac{1}{m}\sum_{i=1}^{m}\langle {}^i\theta(f,.) - H(\theta)(f), S_T(f)\rangle \overset{m\to\infty}{\longrightarrow} 0 \quad \mathbb{B}\text{-a.s.} \tag{41}$$

and by the SLLN one must have $\mathbb{F}$-a.s.

$$\langle H(\theta)(f), S_T(f)\rangle = \langle \mathbf{E}_{\mathbb{B}}[\theta(f,.)], S_T(f)\rangle. \tag{42}$$

Assumption 6 (26) and condition (NA) in M^F imply $\pi_t(H(\theta)) = \pi_t(\mathbf{E}_{\mathbb{B}}[\theta])$ $\mathbb{P}$-a.s. for $t\in\mathbb{T}$. Lemma 5 completes the proof. □

Proof of Theorem 1. From Lemma 4 one has that $\mathbb{Q}\otimes\mathbb{B} \sim \mathbb{F}\otimes\mathbb{B}$. Analogously to Lemma 5, one obtains that (30) exists. Hence, (30) fulfills Assumption 6 (cf. Remark 4 (8)). Furthermore, (30) is a minimum fair price in the sense of Assumption 7 since with $H = H^*$ one has (27) from

$$\mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\langle\theta, S_T\rangle/S_T^0] = \mathbf{E}_{\mathbb{Q}}[\langle H^*(\theta), S_T\rangle/S_T^0] \tag{43}$$

by Fubini's Theorem, and Lemma 5 shows that (i), (ii) and (iii) are fulfilled. Observe that (30) is a valuation principle since $(S_t/S_t^0)_{t\in\mathbb{T}}$ is a $\mathbb{Q}\otimes\mathbb{B}$-martingale and therefore $\pi_t(\theta_t) = \langle\theta_t, S_t\rangle$ for any t-portfolio $\theta_t \in \Theta$ (cf. Remark 4 and Definition 1). Now, uniqueness will be shown. Suppose that π is a minimum fair price in the sense of Assumption 7 and consider some $\{{}^i\theta,\ i \in \mathbb{N}^+\}_{B-i.i.d.}$. Then it is know from Lemma 7 that $\pi_0({}^i\theta) = \pi_0(H^*({}^i\theta)) = \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\langle{}^i\theta, S_T\rangle/S_T^0]$ for all $i \in \mathbb{N}^+$. However, one can choose the set $\{{}^i\theta,\ i \in \mathbb{N}^+\}_{B-i.i.d.}$ such that ${}^1\theta = (\mathbf{1}_Z, 0, \ldots, 0)$, where $\mathbf{1}_Z$ is the indicator function of a cylinder set $Z = F' \times B_1 \times B_2 \times \ldots$ with $F' \in \mathcal{F}_T$ and $B_j \in \mathcal{B}_T^j$ for $j \in \mathbb{N}^+$, where $B_j \neq B^j$ for only finitely many j (Assumption 4 is crucial for the possibility of this choice!). Clearly, these cylinders form a $\cap$-stable generator for $\mathcal{M}_T$, the σ-algebra of the product space, and M itself is an element of this generator. One obtains $\pi_0({}^1\theta) = \mathbb{Q}\otimes\mathbb{B}(Z) = \mathbb{M}(Z)$ from (36) and (25). $\mathbb{M} = \mathbb{Q}\otimes\mathbb{B}$ follows from the coincidence of the measures on the generator. □

Assumptions 6 and 7 could be interpreted as a strong no-arbitrage principle that fulfills (NA) and also excludes arbitrage-like strategies that have their origin in the Law of Large Numbers and the possibility of diversification.

Example 2 (Asymptotic arbitrage opportunities). Consider a set $\{{}^i\theta, i \in \mathbb{N}^+\}_{B-i.i.d.}$ of portfolios. The minimum fair price for each portfolio is given by (30) ($t = 0$). If an insurance company sells the products $\{{}^1\theta, \ldots, {}^m\theta\}$ at that prices, it can buy hedging portfolios such that the mean balance converges to zero almost surely with m (cf. Assumption 7, (*iii*)). However, if the company charges $\pi_0({}^i\theta) + \epsilon$, where $\epsilon > 0$ is an additional fee and π is as in (30), there still is the hedge as explained above, but the gain ϵ per contract was made at $t = 0$. Hence, the safety load ϵ lets the insurance company become a money making machine in the limit. A similar remark can be found in Møller and Steffensen (1994).

So-called asymptotic arbitrage in large markets was originally analyzed in technically very sophisticated papers of Kabanov and Kramkov (1994, 1998). In a paper of Björk and Näslund (1998) on the same topic, there is a relatively easy proof provided for the proposition that the existence of an EMM implies absence of asymptotic arbitrage as defined by them. It seems to be straightforward that our example is covered by their definition (applied to the discrete time case), and hence the existence of the EMM $\mathbb{Q}\otimes\mathbb{B}$ excludes also more general kinds of arbitrage (in Björk's and Näslund's sense) than the simple one given in the example above. Roughly speaking, in an idealized economy close to equilibrium, any EMM $\mathbb{M}'$ of the market $M^{F\times B}$ obtained (indirectly, by the prices) from free trading of portfolios in $M^{F\times B}$ should be expected to be close to $\mathbb{Q}\otimes\mathbb{B}$, where $\mathbb{Q}$ would be the (equilibrium) EMM obtained from just trading in M^F (see also Brennan and Schwartz, 1976). Any strong systematic deviation could give rise to arbitrage-like trading opportunities, as we have just seen.

Remark 7 (Quadratic hedging). Consider an L^2-framework, i.e. the payoff $\langle\theta_t, S_t\rangle$ of any considered t-portfolio θ_t lies in $L^2(M, \mathcal{M}_t, \mathbb{P})$. As $\mathbb{P} = \mathbb{F}\otimes\mathbb{B}$, it can easily be shown that $\mathbf{E}_{\mathbb{B}}[.]$ is the orthogonal projection of $L^2(M, \mathcal{M}_t, \mathbb{P})$ onto its purely financial (and closed) subspace $L^2(F, \mathcal{F}_t, \mathbb{F})$. Standard Hilbert space theory implies that the payoff $\langle\mathbf{E}_{\mathbb{B}}[\theta_t], S_t\rangle = \mathbf{E}_{\mathbb{B}}[\langle\theta_t, S_t\rangle]$ of the hedge $H^*(\theta_t)$ is the best L^2-approximation of the payoff $\langle\theta_t, S_t\rangle$ of the t-portfolio θ_t by a purely financial portfolio in M^F. Furthermore, it can easily be shown that $\mathbb{M} = \mathbb{Q}\otimes\mathbb{B}$ minimizes $||d\mathbb{M}/d\mathbb{P} - 1||_2$ under the constraint $\mathbf{E}_{\mathbb{B}}[d\mathbb{M}/d\mathbb{P}] = d\mathbb{Q}/d\mathbb{F}$ which is implied by Assumption 6. Under some additional technical assumptions, this property is a characterization of the so-called *minimal martingale measure* in the continuous time case (cf. Schweizer (1995b), Møller (2001)). Hence, $\mathbb{Q}\otimes\mathbb{B}$ can be interpreted as the EMM which lies "next" to $\mathbb{P} = \mathbb{F}\otimes\mathbb{B}$ with respect to the L^2-metric. Besides the convergence properties discussed in this paper, these are the most important and "natural" reasons for the use of (30). The hedging method H^* considered here is not the so-called *mean-variance hedge* as it is known from the literature (cf. Bouleau and Lamberton (1989), Duffie and Richardson (1991)). The difference is that the mean-variance approach generally allows for *all* self-financing trading strategies in $M^{F\times B}$, i.e. also biometric events could influence the strategy in this case. However, the ideas are quite similar. An overview concerning hedging approaches in insurance can be found in Møller (2002).

7 Hedging and diversification

In this section, it is shown in which way a life insurance company can hedge its risk by products of the financial market - proposed the market is liquid enough. The technical assumptions are quite weak.

Suppose Assumption 1 to 4 and a set of life policies $\{({}^i\gamma_t, {}^i\delta_t)_{t\in\mathbb{T}} : i \in \mathbb{N}^+\}$ with $\{{}^i\gamma_t : i \in \mathbb{N}^+\}_K$ and $\{{}^i\delta_t : i \in \mathbb{N}^+\}_K$ for all $t \in \mathbb{T}$. Following hedging method H^* of Lemma 5, the portfolios (or strategies replicating) $\mathbf{E}_{\mathbb{B}}[{}^i\gamma_t]$ and $-\mathbf{E}_{\mathbb{B}}[{}^i\delta_t]$ are bought at time 0 for all $i \in \mathbb{N}^+$ and all $t \in \mathbb{T}$. Consider the balance of wins and losses at any time $t \in \mathbb{T}$. For the *mean total payoff per contract at time t* we have

$$\frac{1}{m}\sum_{i=1}^{m}\langle {}^i\delta_t - {}^i\gamma_t - \mathbf{E}_{\mathbb{B}}[{}^i\delta_t - {}^i\gamma_t], S_t\rangle \overset{m\to\infty}{\longrightarrow} 0 \quad \mathbb{P}\text{-a.s.} \tag{44}$$

by Lemma 5. In analogy to Section 5, also the mean *final* balance converges to zero a.s., i.e.

$$\frac{1}{m}\sum_{i=1}^{m}\sum_{t=0}^{T}\langle {}^i\delta_t - {}^i\gamma_t - \mathbf{E}_{\mathbb{B}}[{}^i\delta_t - {}^i\gamma_t], S_T\rangle \overset{m\to\infty}{\longrightarrow} 0 \quad \mathbb{P}\text{-a.s.} \tag{45}$$

This kind of risk management is static in the sense that no trading strategy reacts on biometric events happening after time 0. It corresponds to the considerations in the classical case (Section 5). In Remark 7, it has already been mentioned that the considered hedging method is not the so-called mean-variance hedging. Another more comprehensive but not self-financing hedging approach are the so-called *risk-minimizing strategies* (e.g. Møller (1998, 2001)).

Remark 8. Lemma 6 implies that any of the balances in (44) and (45) has expectation 0 under the physical measure $\mathbb{P} = \mathbb{F} \otimes \mathbb{B}$.

Premium calculation has not yet played any role in this section. However, if the Principle of Equivalence (11) is applied under the minimum fair price (30), one obtains for all $i \in \mathbb{N}^+$

$$\sum_{t=0}^{T}\pi_0(\mathbf{E}_{\mathbb{B}}[-{}^i\delta_t + {}^i\gamma_t]) = \sum_{t=0}^{T}\pi_0({}^i\delta_t - {}^i\gamma_t) = 0. \tag{46}$$

Remark 9. Under (11) and (30), a life office can without any costs at time 0 (!) pursue a self-financing trading strategy such that the mean balance per contract at any time t converges to zero almost surely for an increasing number of individual policies. This is how *diversification* of biometric risks should be understood in our model. The realization of such a hedge would demand the precise knowledge of the second order base given by the Assumptions 1 to 4 (see also Remark 2).

In contrast to other, more comprehensive hedging methods, the presented method has the advantage that there is no need for the risk manager to take into account the biometric development of each individual. The information available at the time of underwriting ($t = 0$) is sufficient, and all strategies are self-financing.

Example 3 (Traditional policies). Consider a life insurance policy which is for the i-th individual given by two cash flows $({}^i\gamma_t)_{t\in\mathbb{T}} = (\frac{{}^iC_t}{S_t^0}e_0)_{t\in\mathbb{T}}$ and $({}^i\delta_t)_{t\in\mathbb{T}} = (\frac{{}^iD_t}{S_t^0}e_0)_{t\in\mathbb{T}}$ with $\mathbb{T} = \{0, 1, \ldots, T\}$ in years. Assume that ${}^i\gamma_t = {}^i\delta_t = 0$ for t greater than some $T_i \in \mathbb{T}$, i.e. the contract has an expiration date T_i, and that each ${}^iC_t$ is for $t \le T_i$ given by ${}^iC_t(f,b) = {}^ic\,{}^i\beta_t^\gamma(b^i)$ for all $(f,b) = (f, b^1, b^2, \ldots) \in M$ where ${}^ic$ is a positive constant. Let $({}^i\delta_t)_{t\in\mathbb{T}}$ be defined analogously with the variables ${}^iD_t, {}^id$ and ${}^i\beta_t^\delta$. Suppose that ${}^i\beta_t^{\gamma(\delta)}$ is $\mathcal{B}_t^i$-measurable with ${}^i\beta_t^{\gamma(\delta)}(b^i) \in \{0,1\}$ for all $b^i \in B^i$ ($t \le T_i$). For the following have in mind that the portfolio e_0/S_t^0 can be interpreted as the guaranteed payoff of one currency unit at time t. This kind of contract is called a *zero-coupon bond with maturity* t and its price at time $s < t$ is denoted by

$p(s, t-s) = \pi_s(e_0/S_t^0)$ where $t-s$ is the time to maturity and $p(s,0) := 1$ for all $s \in \mathbb{T}$.
1. Term assurance. Suppose that for $t \le T_i$ one has ${}^i\beta_t^\gamma = 1$ if and only if the i-th individual has died in $(t-1, t]$ and for $t < T_i$ that ${}^i\beta_t^\delta = 1$ if and only if the i-th individual is still alive at t, but ${}^i\beta_{T_i}^\delta \equiv 0$. Assume that i is alive at $t = 0$. Clearly, this contract is a term assurance with level annual premium ${}^id$ and death benefit ${}^ic$. $\mathbf{E}_\mathbb{B}[{}^i\beta_t^\gamma]$ and $\mathbf{E}_\mathbb{B}[{}^i\beta_t^\delta]$ are mortality, respectively survival probabilities. Respective data can be obtained from mortality tables. The international actuarial notation is ${}_{t-1|1}q_x = \mathbf{E}_\mathbb{B}[{}^i\beta_t^\gamma]$ $(t > 0)$ and ${}_tp_x = \mathbf{E}_\mathbb{B}[{}^i\beta_t^\delta]$ $(0 < t < T_i)$ for an individual of age x (cf. Gerber (1997); for convenience reasons, the notation ${}_{-1|1}q_x = 0$ and ${}_0p_x = 1$ is used in the following). The hedge H^* for ${}^i\delta_t - {}^i\gamma_t$ is for $t < T_i$ given by the number of $({}^ic\, {}_{t-1|1}q_x - {}^id\, {}_tp_x)$ zero-coupon bonds with maturity t, and for $t = T_i$ by ${}^ic\, {}_{T_i-1|1}q_x$ zero-coupon bonds with maturity T_i.
2. Endowment assurance. Assume for $t < T_i$ that ${}^i\beta_t^\gamma = 1$ if and only if the i-th individual has died in $(t-1, t]$, but ${}^i\beta_{T_i}^\gamma = 1$ if and only if i has died in $(T_i - 1, T_i]$ or is still alive at T_i. Furthermore, ${}^i\beta_t^\delta = 1$ if and only if the i-th individual is still alive at $t < T_i$, but ${}^i\beta_{T_i}^\delta \equiv 0$. Assume that i is alive at $t = 0$. This contract is a so-called endowment that features level annual premiums ${}^id$, a death benefit of ${}^ic$ and survival benefit ${}^ic$. The hedge H^* with respect to ${}^i\delta_t - {}^i\gamma_t$ is for $t < T_i$ given by the number of $({}^ic\, {}_{t-1|1}q_x - {}^id\, {}_tp_x)$ zero-coupon bonds with maturity t, and for $t = T_i$ by ${}^ic\,({}_{T_i-1|1}q_x + {}_{T_i}p_x)$ zero-coupon bonds with maturity T_i.

In fact, in the case of traditional contracts, all hedging can be done by zero-coupon bonds (also called *matching*).

Example 4 (Unit-linked products). The case of a unit-linked product is interesting if and only if the product is not the sum of a traditional policy and a simple fund policy (which is sometimes the case in practice). So, let us assume that the policy is given by a cash flow of level premiums $({}^i\delta_t)_{t\in\mathbb{T}}$ as in Example 3 and a flow of benefits $({}^i\gamma_t)_{t\in\mathbb{T}}$ such that ${}^i\gamma_t(f, b) = {}^i\theta_t \cdot {}^ic\, {}^i\beta_t^\gamma(b^i)$ for all $(f, b) \in M$ where ${}^i\theta_t \in \Theta^F$ is an arbitrary purely financial t-portfolio and all other notations are the same as in the introduction of Example 3. For instance, one could consider a number of shares of an index, or a number of assets together with the respective European Puts which ensure a certain level of benefit (i.e. a “unit-linked product with guarantee”). The strategy with respect to ${}^i\delta_t - {}^i\gamma_t$ is given by ${}^ic \cdot \mathbf{E}_\mathbb{B}[{}^i\beta_t^\gamma]$ times the replicating strategy of ${}^i\theta_t$ minus $({}^id \cdot \mathbf{E}_\mathbb{B}[{}^i\beta_t^\delta])$ zero-coupon bonds maturing at time t. In particular, for ${}^i\theta_t$ being a constant portfolio, the strategy is obviously very simple as the portfolio must not be replicated, but can be bought directly.

Remark 10. The technical assumption (K), which is sufficient for the convergence of (44) (cf. Definition 3 (iii)) and which is assumed at the very beginning of this section, will be discussed now. In the case of traditional policies as in Example 3, the realistic condition ${}^ic, {}^id \le const \in \mathbb{R}^+$ for all $i \in \mathbb{N}^+$ implies

(K) for the sets $\{{}^i\gamma_t : i \in \mathbb{N}^+\}$ and $\{{}^i\delta_t : i \in \mathbb{N}^+\}$ for all $t \in \mathbb{T}$. In the case of unit-linked products, suppose that there are only finitely many possible portfolios ${}^i\theta_t$ for each $t \in \mathbb{T}$, which is also quite realistic as often shares of one single fund are considered. Under this assumption, again ${}^ic, {}^id \leq const \in \mathbb{R}^+$ for all $i \in \mathbb{N}^+$ implies (K) for the sets $\{{}^i\gamma_t : i \in \mathbb{N}^+\}$ and $\{{}^i\delta_t : i \in \mathbb{N}^+\}$ for all $t \in \mathbb{T}$. Hence, (K) is no drawback for practical purposes.

8 Comparison with other approaches

A recent version of Møller and Steffensen (1994) describes a diversification approach by claiming that the property that the relative net loss of a portfolio of unit-linked contracts converges to zero with increasing size uniquely characterizes the premium given by the product measure principle. They also point out the possibility of an infinitely large surplus if premiums are taken larger (cf. Example 2). At the stage of this remark, the mathematical framework is not precisely specified and proofs are not provided. Nonetheless, what we have shown in this paper is quite close to what Møller and Steffensen have sketched in their lecture notes. However, it should be pointed out that their loss (balance), in contrast to ours, is discounted to time zero.

The earlier mentioned work of Becherer (2003) considers a utility-indifference approach with respect to exponential utility for valuation and hedging of integrate tradable (e.g. financial) and non-tradable (e.g. biometric) risks. As Becherer points out, this approach can be seen as an adaption of the exponential premium principle to a model with dynamic financial markets. The first half of the article is dedicated to considerations and results for a general semi-martingale market framework. Aiming for more constructive results, Becherer (2003) examines a class of "semi-complete product models" existing of a complete financial sub-market and of an additional countable number of independent (non-tradable) sources of risk. Showing an additivity result for the price of claims that are conditionally (on the financial sub-market) independent, Becherer proves, under certain technical conditions, that the utility-indifference price of an average portfolio (actually: arithmetic mean) of claims, which are i.i.d. conditioned on the financial sub-market, converges to what we call the minimum fair price (by the product measure principle) of such a claim (Theorem 4.11 in Becherer (2003)). An example for a certain type of bounded equity-linked policies for i.i.d. lives is given.

The main difference between Becherer's work and the approach of this paper when deriving the valuation principle is that Becherer assumes the existence of a utility function, whereas we demanded convergence of balances (cf. Assumption 7 (iii)). Another difference is that we showed how diversification in the sense of converging mean balances of hedges and policy cash flows appears, in contrast to Becherer (2003), where a convergence property of the utility-indifference *price* is shown (in general, a full indifference price process is derived). The utility-indifference price at time zero can therefore be seen as an approximation of the minimum fair price in the case of a large portfolio of small contracts, or vice versa. Becherer works mostly in continuous time. Technically, he does not consider a product space, but an original σ-algebra (σ-field) of the financial market augmented by independent ones representing non-tradable risks. Becherer's assumptions about spaces and the time axis are therefore more general than in our setup.

In a quite general continuous time framework, Steffensen (2000) derives a stochastic version of Thiele's Differential Equation, and also the set of possible equivalent martingale measures for the assumed market model. Steffensen's model is much more general as e.g. Aase and Persson (1994), which works basically with a Black-Scholes model. For instance, Steffensen's work also allows for jumps in price processes, and actually his model allows for the trading of mortality risks. In this sense, Steffensen (2000) is much more general than our approach since we just derive *one* (therefore unique) EMM of many possible ones. We restrict the set of possible EMMs by the additional requirement of diversification (convergence property). In Steffensen (2000), the product measure principle is just one of many possible valuation principles which could arise from arbitrage-free trading of insurance products.

The last comparison in this section is regarding an approach which was proposed by an unknown referee. To simplify notation, we will work directly with cash flows instead of portfolios and assume the money account to be constant 1, i.e. $S_t^0 = 1$ for all $t \in \mathbb{T}$. Now, assume B-i.i.d. payoffs H^i ($i \in \mathbb{N}$). Under certain assumptions, the SLLN implies

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} H^i = \mathbf{E}_{\mathbb{F}\otimes\mathbb{B}}[H^1|\mathcal{F}_T \otimes \mathcal{B}_0] := H. \tag{47}$$

Now assume that a valuation principle π is given by some EMM $\mathbb{M}$ which is fair in the sense that

$$\pi_0(H) = \mathbf{E}_{\mathbb{M}}[H] = \mathbf{E}_{\mathbb{Q}}[H], \tag{48}$$

and

$$\pi_0(H^i) = \mathbf{E}_{\mathbb{M}}[H^i] = \mathbf{E}_{\mathbb{M}}[H^j] = \pi_0(H^j) \quad \text{for all } i, j \in \mathbb{N}. \tag{49}$$

From this and the linearity of the expectation operator, we obtain

$$\lim_{m\to\infty} \mathbf{E}_{\mathbb{M}}\left[\frac{1}{m}\sum_{i=1}^{m} H^i\right] = \lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^{m} \mathbf{E}_{\mathbb{M}}[H^i] = \mathbf{E}_{\mathbb{M}}[H^j] \quad \text{for all } j \in \mathbb{N}. \tag{50}$$

The referee now conludes that under sufficient integrability conditions

$$\begin{aligned} \pi_0(H^j) = \mathbf{E}_{\mathbb{M}}[H^j] &= \lim_{m\to\infty} \mathbf{E}_{\mathbb{M}}\left[\frac{1}{m}\sum_{i=1}^{m} H^i\right] \\ &= \mathbf{E}_{\mathbb{M}}\left[\lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^{m} H^i\right] = \mathbf{E}_{\mathbb{M}}[H] = \pi_0(H). \end{aligned} \tag{51}$$

So, roughly speaking, this approach shows quite directly that we are forced to evaluate like $\pi_0(H^i) = \pi_0(\mathbf{E}_{\mathbb{F}\otimes\mathbb{B}}[H^i|\mathcal{F}_T \otimes \mathcal{B}_0]) = \pi_0(\mathbf{E}_{\mathbb{B}}[H^i])$, which is a statement about a valuation principle (and not a measure), and closely related to our essential Lemma 7, seen together with Assumption 7, Eq. (27).

From a technical point of view, the above sketched approach is not much simpler than ours. The reason is that still uniqueness of the EMM has to be proven, taking into account condition (48) for $\mathbb{M}$. A proof would be essentially the same as the one for Theorem 1. Furthermore, integrability conditions must be given for (51), which can be done using the conditions for the Dominated Convergence Theorem. Appropriate would be e.g. our condition (K). The B-i.i.d condition together with Lemma 3 and the SLLN would prove (47). Taking into account all technical subtleties, the amount of work seems to be quite similar for both approaches, perhaps a little less for the one presented above. The main difference, however, is that the above approach does not need a postulation of hedges which let mean balances converge to zero. Instead, it uses the SLLN directly for the claims (cf. Eq. (47)) and, by Dominated Convergence, a property of the Lebesgue integral to derive (51).

9 Incomplete financial markets

Until now, the theory presented in this paper assumed complete and arbitrage-free markets (cf. Assumption 2), which reduces the number of explicit market models that can be considered. However, some of the concepts work, under some restrictions, with incomplete market models.

In particular, it is now assumed that in Assumption 2 completeness of the market model M^F and uniqueness of the EMM $\mathbb{Q}$ is *not* demanded, but $\mathbb{Q} \sim \mathbb{F}$ and $d\mathbb{Q}/d\mathbb{F}$ bounded. Let us enumerate the altered assumption by 2' and define

$$\Theta^F = \{\theta : \theta \text{ replicable by a self-financing strategy in } M^F\} \tag{52}$$

$$\Theta = \{\theta : \theta \in (L^1(M, \mathcal{M}_T, \mathbb{P}))^d \text{ and } \mathbf{E}_{\mathbb{B}}[\theta] \in \Theta^F\}. \tag{53}$$

It is well-known from the theory of financial markets that *any* EMM $\mathbb{Q}$ fulfills pricing formula (3) for any replicable portfolio $\theta \in \Theta^F$. Now, with Θ^F and Θ as defined above and Assumption 2 replaced by 2', it can easily be checked that the Lemmas 4 - 7 still hold. Concerning Theorem 1, π as defined in (30) is for *any* financial EMM $\mathbb{Q}$ a minimum fair price. Hence, uniqueness seems to be lost. However, for any minimum fair price one still has that π_0 is unique on (53). The reason is that for any $\theta \in \Theta$ and any two EMM $\mathbb{Q}$ and $\underline{\mathbb{Q}}$ of M^F

$$\mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\langle\theta, S_T\rangle / S_T^0] = \mathbf{E}_{\underline{\mathbb{Q}}\otimes\mathbb{B}}[\langle\theta, S_T\rangle / S_T^0] \tag{54}$$

by Fubini's Theorem and the (NA)-condition. Hence, pricing at time $t = 0$ and hedging (cf. Section 7) still work as in the case of complete financial markets.

In the presence of arbitrage opportunities, the existence of an equivalent martingale measure gets lost. Nonetheless, assume a financial market model M^F which is neither necessarily arbitrage-free, nor complete and suppose that there is a valuation principle π^F used in M^F on a set Θ^F of purely financial portfolios which are taken into consideration (this does not mean absence of arbitrage). Under the considered Θ^F, define Θ by (53) and for any $\theta \in \Theta$

$$\pi_0(\theta) = \pi_0^F(\mathbf{E}_{\mathbb{B}}[\theta]), \tag{55}$$

which is the price of the hedge H^* at time 0 (compare with (27) and (36) for t = 0). In an L^2-framework as in Remark 7, i.e. if we have for any t that $\langle \Theta, S_t \rangle \subset L^2(M, \mathcal{M}_T, \mathbb{P})$, $\mathbf{E}_{\mathbb{B}}[\theta]$ is the best approximation in Θ^F to any $\theta \in \Theta$ in the L^2-sense (cf. Remark 7). Even if we do not assume the L^2-framework, the properties (i), (ii) and (iii) of Assumption 7 are still fulfilled for the above defined Θ and for H^* as in (35). Hence, π_0 satisfies the demand for converging balances as stated in Principle 7 of Section 2 and the expressions (44) and (45) are still valid. For these reasons, (55) is a rather sensible valuation principle.

10 Example with historical data

It is not new to evaluate life insurance policies with real market data. Many examples can be found in the literature (e.g. Koller, 2000). The following example intends to demonstrate the impact of market-based valuation for a particular set of contracts with German market data.

Let us consider the traditional policies as described in Example 3. Applying the Principle of Equivalence (11), we demand

$$\pi_0\left(\sum_{t=0}^{T_i} {}^i c\, {}^i\beta_t^{\gamma} e_0 / S_t^0\right) = \pi_0\left(\sum_{t=0}^{T_i} {}^i d\, {}^i\beta_t^{\delta} e_0 / S_t^0\right). \tag{56}$$

Now, suppose that the minimum fair price π from (30), respectively valuation principle (55), is applied for premium calculation. Clearly,

$$\frac{{}^i d}{{}^i c} = \sum_{t=0}^{T_i} p(0,t) \cdot \mathbf{E}_{\mathbb{B}}[{}^i\beta_t^{\gamma}] \Big/ \sum_{t=0}^{T_i} p(0,t) \cdot \mathbf{E}_{\mathbb{B}}[{}^i\beta_t^{\delta}] \tag{57}$$

where $p(0,t)$ is the price of a zero-coupon bond as defined in Section 7. An important consequence of (57) is that the quotient ${}^i d/{}^i c$ (minimum fair premium/benefit) depends on the zero-coupon bond prices (or yield curve) at time 0. As the term structure of interest rates varies from day to day, this particularly means that ${}^i d/{}^i c$ varies from day to day and therefore depends on the day of underwriting (actually, it depends on the exact time). Insurance companies do not determine prices daily. Hence, in our model, they give rise to financial risks as policies may be over-valued.

Now, assume that any time value is given in fractions of years. The so-called *spot (interest) rate* $R(t,\tau)$ for the time interval $[t, t+\tau]$ is defined by

$$R(t,\tau) = -\frac{\log p(t,\tau)}{\tau}. \tag{58}$$

The *short rate* $r(t)$ at t is defined by $r(t) = \lim_{\tau\to 0} R(t,\tau)$, where the limit is assumed to exist. The *yield curve* at time t is the mapping with $\tau \mapsto R(t,\tau)$ for $\tau > 0$ and $0 \mapsto r(t)$. Figure 5 shows the historical yield structure (i.e. the set of yield curves) of the German debt securities market from September 1972 to April 2003. The 368 values are taken from the end of each month. The maturities' range is 0 to 28 years. The values for $\tau > 0$ were computed via a parametric presentation of yield curves (the so-called Svensson-method; cf. Schich (1997)) for which parameters can be taken from the Internet page of the German Federal Reserve (*Deutsche Bundesbank*; `http://www.bundesbank.de`). The implied Bundesbank values R' are estimates of *discrete* interest rates on notional zero-coupon bonds based on German Federal bonds and treasuries (cf. Schich, 1997) and have to be converted to continuously compounded interest rates (as implicitly used in (58)) by $R = \ln(1 + R')$. As approximation for the short rate, the day-to-day money rates from the Frankfurt market (*Monatsdurchschnitt des Geldmarktsatzes für Tagesgeld am Frankfurter Bankplatz*; also available at the Bundesbank homepage) are taken and converted into continuous rates. Actually, the short rate is not used in the following but completes Figure 5.

Equation (58) shows that interest rates (yields) and zero-coupon bond prices contain the same information, namely the present value of a non-defaultable future payoff. As there is a yield curve given for any time t of the considered historical time axis, it is possible to compute the historical value of ${}^i d/{}^i c$ for t (which is the date of underwriting for the respective contract) via (58) and (57). Doing so, one obtains

$$\frac{{}^i d}{{}^i c}(t) = \sum_{\tau=0}^{T_i} p(t,\tau)\, {}_{\tau-1|1}q_x(t) \Big/ \sum_{\tau=0}^{T_i-1} p(t,\tau)\, {}_{\tau}p_x(t) \tag{59}$$

for the traditional term assurance and

$$\frac{{}^i d}{{}^i c}(t) = \left(p(t,T_i)\, {}_{T_i}p_x(t) + \sum_{\tau=0}^{T_i} p(t,\tau)\, {}_{\tau-1|1}q_x(t) \right) \Big/ \sum_{\tau=0}^{T_i-1} p(t,\tau)\, {}_{\tau}p_x(t) \tag{60}$$

for the endowment (cf. Example 3). The values ${}_{\tau-1|1}q_x$ ($\tau > 0$) and ${}_{\tau}p_x$ ($0 < \tau < T_i$) are taken from (or computed by) the DAV (*Deutsche Aktuarvereinigung*) mortality table "1994 T" (Loebus, 1994), the value ${}_{T_i}p_x$ is computed by the table "1994 R" (Schmithals and Schütz, 1995). The reason for the different tables is that in actuarial practice mortality tables contain safety loads which depend on whether the death of a person is in (financial) favour of the insurance company, or not. In this sense, the used mortality tables are first order tables (cf. Remark 2). Clearly, the use of internal second order tables of real life insurance companies would be more appropriate. However, for competitive reasons they are usually not published. All probabilities mentioned above are considered to be constant in time. Especially, to make things easier, there is no “aging shift” applied to table “1994 R”.

Now, consider a man of age $x = 30$ years and the time axis $\mathbb{T} = \{0, 1, \dots, 10\}$ (in years). In Figure 1, the rescaled quotients (59) and (60) are plotted for the above setup. For comparison reasons: the absolute values at the starting point (September 1972) are ${}^i d/{}^i c = 0.063792$ for the endowment, respectively ${}^i d/{}^i c = 0.001587$ for the term assurance. The plot nicely shows the dynamics of the quotients and hence of the minimum fair premiums ${}^i d$ if the benefit ${}^i c$ is assumed to be constant. The premiums of the endowment seem to be much more subject to interest rate fluctuations than the premiums of the term assurance. For instance, the minimum fair annual premium ${}^i d$ for the 10-year endowment with a benefit of ${}^i c = 100,000$ Euros was 5,285.55 Euros at the 31st July 1974 and 8,072.26 at the 31st January 1999. For the term assurance (with the same benefit), one obtains ${}^i d = 152.46$ Euros at the 31st July 1974 and 168.11 at the 31st January 1999 (cf. Table 1).

If one assumes a discrete technical (= first order) rate of interest R'_{tech}, e.g. 0.035, which is the mean of the interest rates legally guaranteed by German life insurers, one can compute technical quotients ${}^i d_{\text{tech}}/{}^i c$ by computing the technical values of zero-coupon bonds, i.e. $p_{\text{tech}}(t, \tau) = (1 + R'_{\text{tech}})^{-\tau}$, and plugging them into (59), resp. (60). If a life insurance company charges the technical premiums ${}^i d_{\text{tech}}$ instead of the minimum fair premiums ${}^i d$ and if one considers the valuation principle (30), respectively (55), to be a reasonable choice, the *market value* of the considered policy at time t is

$$ {}^i MV = ({}^i d_{\text{tech}} - {}^i d) \cdot \sum_{\tau=0}^{T_i - 1} p(t, \tau) \, {}_\tau p_x(t) \tag{61} $$

due to the Principle of Equivalence, respectively (56). In particular, this means that the insurance company can book the gain or loss (61) in the mean (or limit; cf. Example 2 and Remark 8) at time 0 as long as proper risk management, as described in Section 7, takes place afterwards. Thus, the market value (61) is a measure for the profit, or simply *the expected discounted profit* of the considered contract if one neglects all additional costs and the fact that first order mortality tables are used.

Figure 2 shows the historical development of ${}^i MV/{}^i c$ (market value/benefit) for the 10-year endowment as described above (solid line). For instance, the market value ${}^i MV$ of a 10-year endowment with a benefit of ${}^i c = 100,000$ Euros was 20,398.70 Euros at July 31, 1974. At the 31st January 1999, it was worth 2,578.55 Euros, only. The situation becomes even worse in the case of a technical (or promised) rate of interest $R'_{\text{tech}} = 0.050$ (dashed line) - which is quite little in contrast to formerly promised returns of e.g. German life insurers. At the 31st January 1999, such a contract was worth -3,141.95

Euros, i.e. the contract actually produced a loss in the mean. Some market values of the 10-year term assurance can be found in Table 1 on page 32.

All computations from above have also been carried out for a 25-year endowment, respectively term assurance (cf. Table 1). The corresponding figures are 3 and 4. Concerning Figure 3, the absolute values at the starting point (September 1972) are ${}^i d/{}^i c = 0.013893$ for the endowment, respectively ${}^i d/{}^i c = 0.002553$ for the life assurance. The minimum fair premium ${}^i d$ for the 25-year endowment with benefit ${}^i c = 100,000$ Euros was 808.39 Euros at the 31st July 1974 and 2,177.32 Euros at the 31st January 1999. For the term assurance with the same benefit, one obtains ${}^i d = 216.37$ Euros at the 31st July 1974 and 303.90 at the 31st January 1999. Hence, the premium-to-benefit ratio for both types of contracts seems to be more dependent on the yield structure than in the 10-year case. However, compared to the 10-year contracts, the longer running time seems to stabilize the market values of the contracts (cf. Table 1 and Figure 4). Nonetheless, they are still strongly depending on the yield structure.

11 Conclusion

The paper has shown that the product measure valuation principle (minimum fair price), which is frequently used in modern life insurance mathematics, follows from a set of eight principles, or seven mathematical assumptions, defining the model framework. One of them, diversification, was the demand for converging mean balances under certain, rather rudimentary, hedges which must be able to be financed by the minimum fair prices. As in the classical case, the Law of Large Numbers plays a fundamental role, here. Actually, only two principles, the demand for complete, arbitrage-free financial markets and the principle of no-arbitrage pricing, were in their origin not traditional. The examples in the last section, but also the hedging examples in the sections before, have once more confirmed the importance of market-based valuation principles and financial hedging methods in the modern practice of life insurance mathematics.

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A Figures and tables

Date	1974/07/31	1999/01/31
Term assurance: 10 years		
Techn. premium ${}^i d_{\text{tech}}$ ($R'_{\text{tech}} = 0.035$)	168.94	
Techn. premium ${}^i d_{\text{tech}}$ ($R'_{\text{tech}} = 0.050$)	165.45	
Minimum fair annual premium ${}^i d$	152.46	168.11
Market value ${}^i MV$ ($R'_{\text{tech}} = 0.035$)	108.90	7.17
Market value ${}^i MV$ ($R'_{\text{tech}} = 0.050$)	85.84	-22.80
Term assurance: 25 years		
Techn. premium ${}^i d_{\text{tech}}$ ($R'_{\text{tech}} = 0.035$)	328.02	
Techn. premium ${}^i d_{\text{tech}}$ ($R'_{\text{tech}} = 0.050$)	303.27	
Minimum fair annual premium ${}^i d$	216.37	303.90
Market value ${}^i MV$ ($R'_{\text{tech}} = 0.035$)	1,009.56	376.84
Market value ${}^i MV$ ($R'_{\text{tech}} = 0.050$)	785.80	-9.83
Endowment: 10 years		
Techn. premium ${}^i d_{\text{tech}}$ ($R'_{\text{tech}} = 0.035$)	8,372.65	
Techn. premium ${}^i d_{\text{tech}}$ ($R'_{\text{tech}} = 0.050$)	7,706.24	
Minimum fair annual premium ${}^i d$	5,285.55	8,072.26
Market value ${}^i MV$ ($R'_{\text{tech}} = 0.035$)	20,398.70	2,578.55
Market value ${}^i MV$ ($R'_{\text{tech}} = 0.050$)	15,995.27	-3,141.95
Endowment: 25 years		
Techn. premium ${}^i d_{\text{tech}}$ ($R'_{\text{tech}} = 0.035$)	2,760.85	
Techn. premium ${}^i d_{\text{tech}}$ ($R'_{\text{tech}} = 0.050$)	2,255.93	
Minimum fair annual premium ${}^i d$	808.39	2,177.32
Market value ${}^i MV$ ($R'_{\text{tech}} = 0.035$)	17,655.42	9,118.39
Market value ${}^i MV$ ($R'_{\text{tech}} = 0.050$)	13,089.53	1,228.34

Table 1: Selected (extreme) values for varying policies for a 30 year old man (fixed benefit: ${}^i c = 100,000$ Euros)

Figure 1: Rescaled plot of the quotient ${}^{i}d/{}^{i}c$ (minimum fair annual premium/benefit) for the 10-year endowment (solid), resp. term assurance (dashed), for a 30 year old man

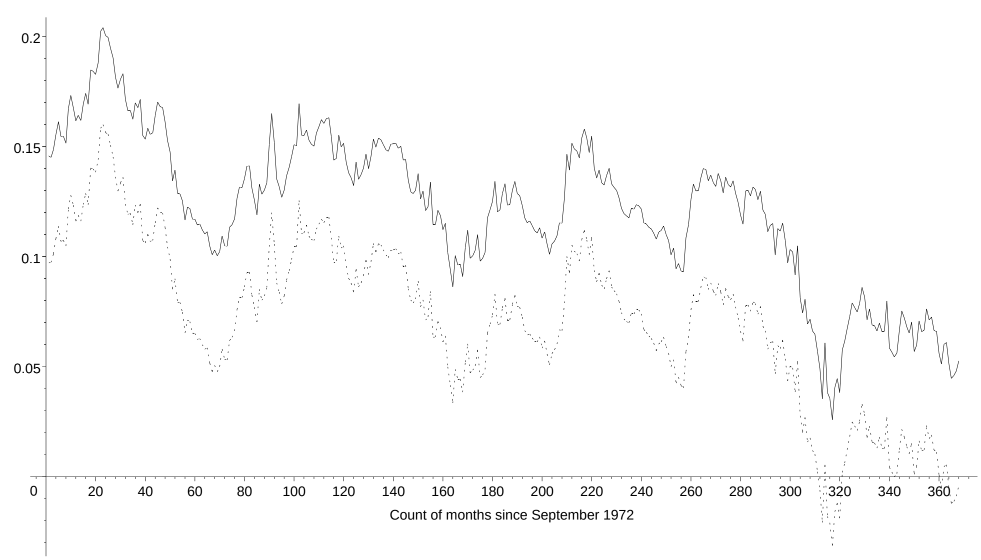

Figure 2: ${}^{i}MV/{}^{i}c$ (market value/benefit) for the 10-year endowment under a technical interest rate of 0.035 (solid) and 0.050 (dashed)

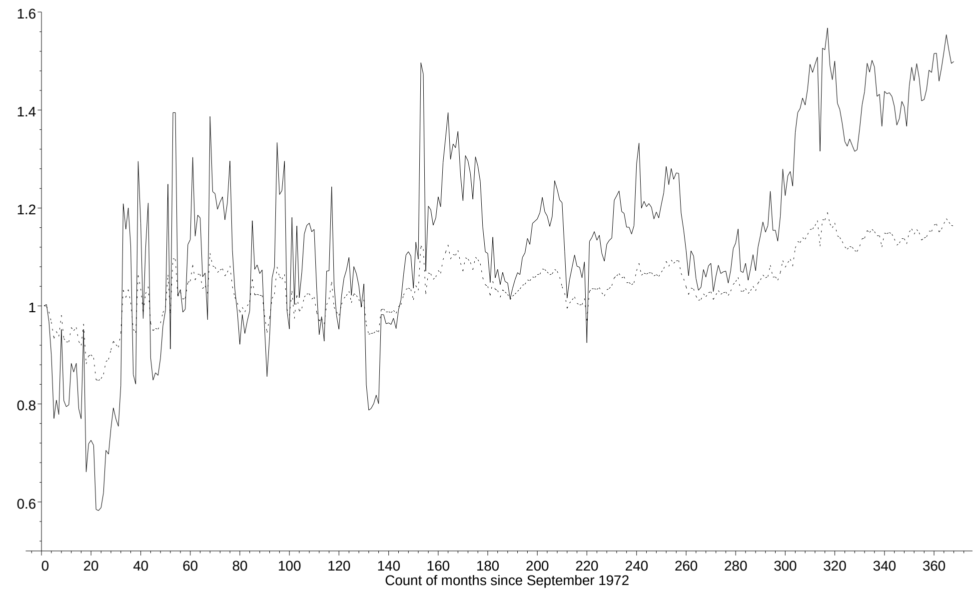

Figure 3: Rescaled plot of the quotient ${}^{i}d/{}^{i}c$ (minimum fair annual premium/benefit) for the 25-year endowment (solid), resp. term assurance (dashed), for a 30 year old man

Figure 4: ${}^{i}MV/{}^{i}c$ (market value/benefit) for the 25-year endowment under a technical interest rate of 0.035 (solid) and 0.050 (dashed)

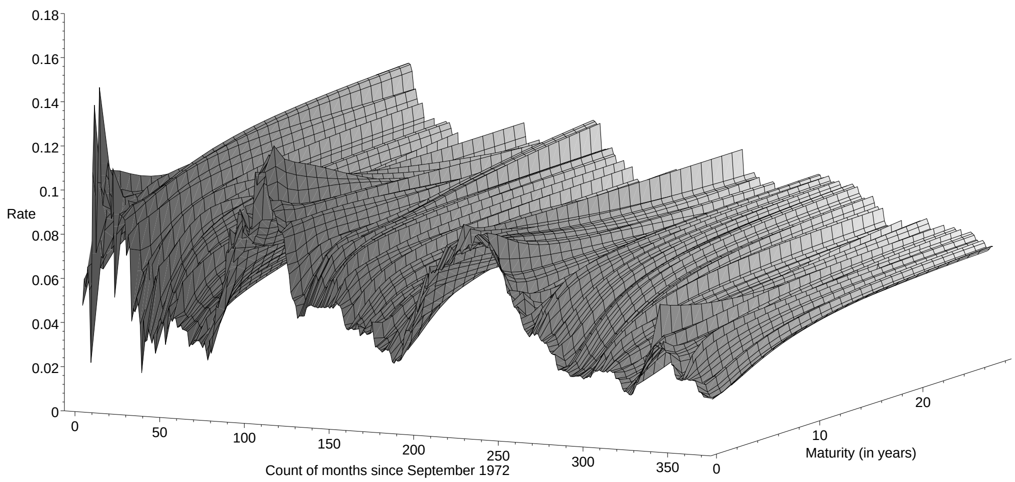

Figure 5: Historical yields of the German debt securities market