Valuation in the structural model of systemic interconnectedness

Tom Fischer

University of Wuerzburg

November 27, 2014

Abstract

This presentation summarizes some previously published and several to-date unpublished results for an asset and derivatives pricing model which accounts for systemic counterparty risk in a structural manner. The model allows for the cross-ownership of equities and liabilities within a network of financial entities. Assets and liabilities within the system, as well as the corresponding ownership structures, are allowed to depend on the prices of system-exogenous assets. Liabilities, which can also be derivatives of system-exogenous or system-endogenous assets, belong to one of potentially many seniority classes, whose order of priority is properly incorporated. The presented work generalizes some of the results by Eisenberg and Noe, Suzuki, Elsinger, and Gouriéroux et al., and can be understood as a far-reaching extension of the Merton model. Of particular concern are the existence and the uniqueness of price solutions, as well as the existence of greatest (i.e. globally Pareto-dominant) solutions when multiple equilibria exist. In the latter case, unambiguous risk-neutral pricing of all liabilities might still be possible. In line with previous results by Elsinger, the value of a system outsider's holdings is always uniquely determined, and – for the group of all system outsiders and bankruptcy costs absent – it never pays to bail out defaulting entities.

Synonyms

- Systemic **Interconnectedness**
- Systemic **Interdependence**
- Systemic **Counterparty Risk**
- **Systemic Risk**
- **Cross-Ownership** ("XOS") of derivatives, debt, equity
- "Everyone can own everyone else's stuff."

Focus:

Asset and derivative pricing under any of the above.

Valuation problems under interconnectedness

- **Chains**:

A influences B influences C influences D.

- **Closed Chains** (Vicious Circles):

B might hold shares of A, C holds some debt of B, D owns a derivative issued by C, and A owns some debt of D.

⇒ The share price of A could influence . . . everything else, including itself!

- This can get **extremely complicated** – e.g. 100 firms, several tranches (seniorities) of debt, numerous derivatives written on various assets within/outside the system.

⇒ **No** price equilibria might exist.

⇒ **Multiple** price equilibria might exist.

Uniqueness and existence (at maturity)?

Risk-neutral pricing of all assets & derivatives in the system?

Pricing under non-uniqueness?

Example: Multiple equilibria

Issuers of binary options:

- $n > 1$ firms, to be liquidated at $T > 0$.
- $a_i \geq 0$ ... (stochastic) value of business assets of firm $i = 1, \dots, n$ in T.
- $d_i \geq 0$ ... nominal value of zero-coupon debt (due in T).
- Firm $i > 1$ has issued a binary option written on the equity of firm $i - 1$.
- Firm 1 has issued one on the equity of firm n (**closed chain**).
- B_i payable for the binary option written on firm i.

⇒ **Equities**:

$$\begin{aligned} e_1 &= (a_1 - d_1 - B_n)^+, \\ e_i &= (a_i - d_i - B_{i-1})^+ \qquad \text{for } i > 1. \end{aligned} \tag{1}$$

- For constants $\overline{e}_i \geq 0$ and $b_i > 0$, define

$$B_i = b_i \mathbf{1}_{\{e_i \leq \overline{e}_i\}}. \tag{2}$$

⇒ b_i is paid if the equity of firm i reaches/falls below $\overline{e}_i$.

Example: Multiple equilibria (cont.)

- Consider market scenario with (prob. > 0 for stochastic a_i)

$$\begin{aligned} \overline{e}_i &< a_i - d_i \leq \overline{e}_i + b_{i-1} \quad \text{for } i > 1, && (3) \\ \overline{e}_1 &< a_1 - d_1 \leq \overline{e}_1 + b_n. && (4) \end{aligned}$$

- Recall

$$\begin{aligned} e_1 &= (a_1 - d_1 - B_n)^+, && (5) \\ e_i &= (a_i - d_i - B_{i-1})^+ \quad \text{for } i > 1, && (6) \\ B_i &= b_i \mathbf{1}_{\{e_i \leq \overline{e}_i\}}. && (7) \end{aligned}$$

⇒ Exactly **two possible equilibria**:
(1) all firms have triggered the corresponding binary options, or
(2) no firm has.

⇒ Two solutions for the equities – but which one is ‘right’ or ‘optimal’ for everyone involved? Here: equity/debt holders $\overset{vs.}{\longleftrightarrow}$ option holders.

- Examples, that are more realistic and much less obvious, exist.

⇒ **No-arbitrage valuation can fail in arbitrage-free complete markets!**

General model: Setup

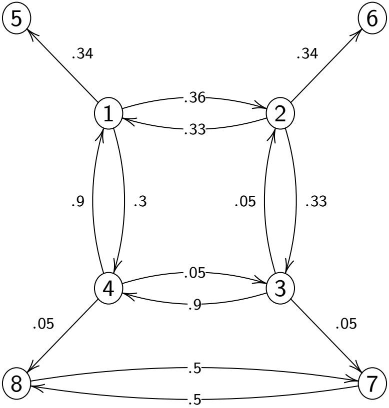
- Firms $1, \ldots, n$ ("system"); time horizon $T \geq 0$.
- Exogenous assets $\mathbf{a} \in A \subset (\mathbb{R}_0^+)^q$.
  Given price vector for q assets, independent of system structure.
- Each firm can owe up to $m+1$ liabilities with seniorities $0, \ldots, m$:
  0 the lowest (= equity), m the highest.
- Nominal value of i's seniority-k liability is d_i^k ($k > 0$).
- Recovery value $0 \leq r_i^k \leq d_i^k$ will actually be paid by firm i.
- Seniority-k liabilities for $k = 1, \ldots, m$, defined by

$$\begin{aligned} \mathbf{d}_{\mathbf{a}}^k : \mathbb{R}^{n(m+1)} &\longrightarrow (\mathbb{R}_0^+)^n \\ \begin{pmatrix} \mathbf{r}^m \\ \vdots \\ \mathbf{r}^0 \end{pmatrix} &\longmapsto \mathbf{d}^k_{\mathbf{r}^m,\ldots,\mathbf{r}^0,\mathbf{a}} = \begin{pmatrix} d_1^k(\mathbf{r}^m, \ldots, \mathbf{r}^0, \mathbf{a}) \\ \vdots \\ d_n^k(\mathbf{r}^m, \ldots, \mathbf{r}^0, \mathbf{a}) \end{pmatrix}, \end{aligned} \tag{8}$$

  $\mathbf{r}^k$ = corresponding vectors of recovery claims.
  - E.g., $d_i^k = 1,000,000$ would be zero-coupon debt.
  - $d_i^k = (a_j - K)^+$ would be a European call on exogenous asset j.
  - $d_i^k = (K - r_j^0)^+$ would be a European put on firm j's equity.

General model: Ownership structures

Who **owes** what? ✓ But: Who **owns** what?



Ownership structure= weighted directed graph.

E.g.: How much of firm 1's seniority-k liability is owned by firm 4? (30%)

General model: Ownership matrices

Following ownership matrix corresponds to the shown structure:

	1	2	3	4	5	6	7	8
1		.33		.9				
2	.36		.05					
3		.33		.05				
4	.3		.9					
5	.34							
6		.34						
7			.05					.5
8				.05			.5	

- Entities in rows own fractions of liabilities of entities in columns.
- Ownership matrices are **left substochastic**.
- Ownership matrices $\mathbf{M}_{\mathbf{a}}^k \in (\mathbb{R}_0^+)^{n \times n}$ for seniority-k liabilities.
- Value of **endogenous assets** (system liabilities) that firm i holds $=$ i-th entry of

$$\sum_{k=0}^{m} \mathbf{M}_{\mathbf{a}}^k \mathbf{r}^k.$$

General model: System equations

- Let $\mathbf{e_a} \in (\mathbb{R}_0^+)^n$ denote the **exogenous assets** held by the n firms.
- E.g., $\mathbf{e_a} = \mathbf{M_a^a}\mathbf{a}$ with $\mathbf{M_a^a} \in (\mathbb{R}_0^+)^{n\times q}$.
- Firm values $= \mathbf{e_a} + \sum_{k=0}^m \mathbf{M_a^k}\mathbf{r}^k$.
- Liquidation value equations (no instantaneous arbitrage) for the system at maturity under the **Absolute Priority Rule**:

$$\mathbf{r}^m = \min\left\{\mathbf{d}^m_{\mathbf{r}^m,\dots,\mathbf{r}^0,\mathbf{a}}, \quad \mathbf{e_a} + \sum_{k=0}^m \mathbf{M_a^k}\mathbf{r}^k\right\} \tag{9}$$

$$\mathbf{r}^j = \min\left\{\mathbf{d}^j_{\mathbf{r}^m,\dots,\mathbf{r}^0,\mathbf{a}}, \left(\mathbf{e_a} + \sum_{k=0}^m \mathbf{M_a^k}\mathbf{r}^k - \sum_{k=j+1}^m \mathbf{d}^k_{\mathbf{r}^m,\dots,\mathbf{r}^0,\mathbf{a}}\right)^+\right\} \tag{10}$$

$$(0 < j < m)$$

$$\mathbf{r}^0 = \left(\mathbf{e_a} + \sum_{k=0}^m \mathbf{M_a^k}\mathbf{r}^k - \sum_{k=1}^m \mathbf{d}^k_{\mathbf{r}^m,\dots,\mathbf{r}^0,\mathbf{a}}\right)^+. \tag{11}$$

Existence and uniqueness of solutions?

General model: Theorem 1

For $\mathbf{a} \in A$, *and if* $||\mathbf{M}_{\mathbf{a}}^k||_1 < 1 \; \forall k$:

1. *The system* (9) – (11) *can only have non-negative solutions.*
2. *For continuous liabilities, the system has at least one solution.*
3. *If* i*'s liabilities* d_i^k *for* $k > 0$ *are monotone increasing and their sum* $\sum_{k=1}^m d_i^k$ *is 1-Lipschitz (non-expansive) in the endogenous assets held by* i, *the solution is unique and* **all endogenous assets are derivatives of the exogenous assets**.

$$\Psi : \mathbf{a} \longmapsto \mathbf{R}^*(\mathbf{a}) = \begin{pmatrix} \mathbf{r}^{m*}(\mathbf{a}) \\ \vdots \\ \mathbf{r}^{0*}(\mathbf{a}) \end{pmatrix} \tag{12}$$

is Lebesgue measurable if $\mathbf{e}_{\mathbf{a}}$, $\mathbf{d}^k_{\mathbf{r}^m,\ldots,\mathbf{r}^0,\mathbf{a}}$ *and* $\mathbf{M}_{\mathbf{a}}^k$ *are measurable.*

Remarks

- Extends results of 2014 Math.Finance publication (publ. online 06/2012).
- Upper bound of the aggregate firm value in terms of $||\mathbf{e_a}||_1$ exists.
- Proof by Brouwer-Schauder and Banach Fixed Point Theorem (contraction w.r.t. ℓ_1-norm).
- Algorithm (Picard Iteration): For any $\mathbf{x} \in \mathbb{R}^{n(m+1)}$,

$$\Psi(\mathbf{a}) = \mathbf{R}^*(\mathbf{a}) = \lim_{l \to \infty} \Phi^l_{\mathbf{a}}(\mathbf{x}), \tag{13}$$

  where Φ is the RHS of (9) – (11).
- $\Psi(\mathbf{a})$ can be used for **simultaneous risk-neutral pricing of all assets and liabilities** in the system at any time before maturity.

$\Rightarrow$ A comprehensive structural model for valuation and (credit/systemic) risk modeling under financial interconnectedness.

- Weaker conditions on ownership matrices?
- Weaker conditions on derivatives?

More results in a few slides, but first:

A short history of structural models of financial interconnectedness

Note: "XOS" stands for "cross-ownership" / "interconnectedness".

1974 **Merton**: *On the pricing of corporate debt: the risk structure of interest rates* (Journal of Finance).

- $n = 1$, $m = 1$, no XOS; equations are trivial.
- Huge impact in theory and practice.

1993 **Kealhofer and Bohn**: Portfolio Management of Default Risk (KMV publication)

- Multi-firm Merton model: n firms, $m = 1$; asset correlations (e.g. $q = n$).
- No XOS (i.e. $\mathbf{M}^0 = \mathbf{M}^1 = \mathbf{0}$); equations are trivial.
- Focus on credit risk.
- Commercially very successful (KMV, now Moody's Analytics).

2001 **Eisenberg and Noe**: *Systemic Risk in Financial Systems* (Management Science).

- n firms, $m = 1$, but $\mathbf{M}^0 = \mathbf{0}$, i.e. XOS of plain vanilla debt only.
- "Fictitious Default Algorithm": finite, no Picard Iteration.
- Focus on clearing systems; no mentioning of Merton model.
- Risk-neutral valuation as an "extension".
- Large impact.

Short history (cont.)

2002 **Suzuki**: *Valuing Corporate Debt: The Effect of Cross-Holdings of Stock and Debt* (Journal of the Operations Research Society of Japan).

- n firms, $m = 1$, XOS of equity and plain vanilla debt.
- Algorithm: Picard Iteration (contraction proof).
- Focus on valuation; knowingly extends Merton (1974).
- Explicit solutions for $n = 2$.
- Unaware of Eisenberg and Noe (2001) (part. rediscovery).
- Widely ignored by scientific community.

2009 **Elsinger**: *Financial Networks, Cross Holdings, and Limited Liability* (Working Paper, Oesterreichische Nationalbank).

- n firms, XOS of equity and multi-seniority plain vanilla debt ($m > 1$).
- No derivatives.
- Algorithm potentially infinite (no Picard Iteration).
- Focus on extension of Eisenberg and Noe (2001).
- Risk-neutral valuation is mentioned, but not the Merton model.
- Unaware of Suzuki (2002) (part. rediscovery).
- Not frequently referred to, but precursor publications (Elsinger, Lehar and Summer 2006a,b) widely known and referred to.

Short history (cont.)

2012/13 **Gouriéroux, Héam and Monfort**: *Bilateral exposures and systemic solvency risk* (Canadian Journal of Economics, 2012); *Liquidation equilibrium with seniority and hidden CDO* (Journal of Banking & Finance, 2013).

- First article partially rediscovers Suzuki's/Elsinger's results without seniorities (unaware of Suzuki (2002), Elsinger (2009), F. (2014)).
- Provides no general algorithm; focus also on contagion.
- Second article aware of Elsinger (2009) with seniority levels, but provides an independent proof for 2 seniority levels; no general derivatives.
- Algorithm: For $\mathbf{M}^0 = \mathbf{0}$, a linear program/simplex method.

2012/14 **Fischer**: *No-Arbitrage Pricing under Systemic Risk: Accounting for Cross-Ownership* (Mathematical Finance 2014, publ. online 06/2012).

- Developed as an extension of Merton (1974).
- Originally unaware of all post-2000 results.
- Eisenberg/Noe and Suzuki mentioned by colleagues/referees.
- Extension of previous results; inclusion of derivatives (multi. sen.).

- W.r.t. existence and uniqueness (ignoring minor differences and except for Gouriéroux et al.), later work extends earlier results (desp. unawareness).
- Post-2000 models need/come with existence and uniqueness results.
- Post-2000 models can be considered as multi-firm Merton models.

General model: More results for weaker matrix properties

- Theorem 1 needed $||\mathbf{M}_{\mathbf{a}}^k||_1 < 1 \; \forall k$ (**strictly** substochastic matrices)

$\Rightarrow$ A fraction of each system liability must be owned by a system outsider.

- Problem: All-encompassing (complete) systems have no outsiders.
- For fixed $\mathbf{a} \in A$, Elsinger (2009) shows that **for constant liabilities**, a weaker condition for all $\mathbf{M}^k$ is sufficient for existence and uniqueness:

Definition: *A left substochastic matrix has the* **Elsinger property** *if there exists no simultaneous permutation of its rows and columns such that the resulting matrix contains a fully left stochastic square submatrix in the upper left corner.*

Note: $||\mathbf{M}||_1 < 1 \Rightarrow$ 'Elsinger', but not the converse. E.g. $\begin{pmatrix} 0.1 & 0.5 \\ 0.1 & 0.5 \end{pmatrix}$.

General model: Globally Pareto-dominant solutions

- Elsinger (2009) shows that if only $\mathbf{M}_\mathbf{a}^0$ has the property, then a greatest and a least solution vector exists (extension of a result by Eisenberg and Noe (2001)).
- **Relevance**: Greatest solution Pareto-dominates all other solutions ($=$ globally Pareto-dominant).

$\Rightarrow$ Everyone interested in getting the most out of the system's liabilities should be happy with this solution!
E.g., within the system, everyone's equity gets maximized.

- Owners of the liabilities should have a say in what solution will be chosen.

$\Rightarrow$ Risk-neutral pricing can still work if multiple equilibria exist, and everyone affected agrees on the greatest solution.

- Binary options example did not have a globally Pareto-dominant solution.

General model: Globally Pareto-dominant solutions (cont.)

Theorem 2 (working paper): *If* $\mathbf{M}_{\mathbf{a}}^0$ *has the Elsinger property, all* $\mathbf{d}_{\mathbf{a}}^k$ *are bounded with respect to* $(\mathbf{r}^m, \ldots, \mathbf{r}^0)$ *and the monotone 1-Lipschitz condition holds, then there exists a measurable non-negative globally Pareto-dominant solution.*

- Picard Iteration ✓ (with explicitly given starting point)
- Risk-neutral pricing ✓
- Proof uses Tarski Fixed Point Theorem.
- Boundedness for given **a** not a strong restriction (e.g. choose 10^{100}).
- There also exists a least non-negative solution.
- $\mathbf{d}_{\mathbf{a}}^k$ only continuous, then at least one non-negative solution exists (Brouwer-Schauder); all non-negative solutions lie in an explicitly given bounded interval.

Corollary: *If* $d_i^k(\mathbf{a})$ *is partially owned by system outsiders, then* $r_i^k(\mathbf{a})$ *is uniquely determined.*

- For system outsiders, price ambiguity under the conditions of Theorem 2 is irrelevant (Elsinger (2009) has this for constant liabilities).

Bailouts: Outsiders' point of view

- Accounting equations

$$\mathbf{e_a} + \sum_{k=0}^{m} \mathbf{M}_\mathbf{a}^k \mathbf{r}^k = \sum_{k=0}^{m} \mathbf{r}^k \quad (14)$$

exo. assets + receivables = equity + payable liabilities

hold for solutions of (9) to (11) (shown in Fischer (2014), they also hold if ownership matrices have ℓ^1-norm 1).

⇒ In aggregate, capital structure and ownership structure are irrelevant to system outsiders, since

$$\underbrace{\sum_{k=0}^{m} \left|\left|\mathbf{r}^k\right|\right|_1 - \sum_{k=0}^{m} \left|\left|\mathbf{M}_\mathbf{a}^k \mathbf{r}^k\right|\right|_1}_{\text{all claims} - \text{internal claims}} = ||\mathbf{e_a}||_1 \quad (15)$$

= (value of) external claims = (value of) exogenous assets.

⇒ Bankruptcy costs absent: **What you put in is what you get out.**

The monotone 1-Lipschitz condition

- Very strong condition:
  An increment in any firm's non-equity liabilities must be super-hedged by the simultaneous increment of its endogenous assets.
- Seems to be difficult to get rid of.
- Derivatives must be well-behaved, cannot be too 'steep'.
- There are much less obvious examples than binary options and derivatives need not have jumps or be large to cause problems:

  Examples with continuous, bounded and 'small' liabilities exist, where risk-neutral **valuation is impossible**.

- Problem: **What is really out there?**

Related work on numerical algorithms

- Eisenberg and Noe's algorithm is finite (XOS of simple debt only).
- Elsinger's algorithm has potentially infinite running time.
- Picard type algorithms (Suzuki, Fischer) have potentially infinite running time.
- If true derivatives are involved, Picard seems to be unavoidable.

Working paper (with J. Hain):

- Without true derivatives (e.g. Eisenberg & Noe, Suzuki, Elsinger, Gouriéroux et al.), algorithms exist that find exact solutions in finite time.
- Three types: Picard, Elsinger, extended Elsinger.
- For large systems, a finite Picard-type algorithm seems to be most efficient (simulation study in work).

A communication problem?

Valuation and systemic risk research are not two separate things.

$\Downarrow$

Need of a **unified model** for

- equity and debt valuation under financial interconnectedness.
- options & derivatives pricing under systemic counterparty risk.
- credit risk management.
- systemic risk research (‘contagion’).
- regulatory purposes (to avoid ‘valuation impossible’).

Further research in the presented model:

- Weaker conditions for derivatives that guarantee uniqueness or a globally Pareto-dominant solution?
- Different liabilities that have the same seniority.
- Multi-period case.

Some literature

- Eisenberg, L., and T. H. Noe (2001): Systemic Risk in Financial Systems. *Management Science* **47** (2), 236-249.
- Elsinger, H. (2009): Financial Networks, Cross Holdings, and Limited Liability. Working Paper 156, Oesterreichische Nationalbank, Wien.
- Fischer, T. (2014): No-Arbitrage Pricing under Systemic Risk: Accounting for Cross-Ownership. *Mathematical Finance* **24** (1), 97–124.
- Gouriéroux, J.-C. Héam and A. Monfort (2012) Bilateral exposures and systemic solvency risk. *Canadian Journal of Economics* **45** (4), 1273–1309.
- Gouriéroux, J.-C. Héam and A. Monfort (2013) Liquidation equilibrium with seniority and hidden CDO. *Journal of Banking & Finance* **37** (2013) 5261–5274.
- Merton, R. C. (1974): On the pricing of corporate debt: the risk structure of interest rates. *Journal of Finance* **29** (2), 449-470.
- Suzuki, T. (2002): Valuing Corporate Debt: The Effect of Cross-Holdings of Stock and Debt. *Journal of the Operations Research Society of Japan* **45** (2), 123-144.

Contact

Prof. Dr. Tom Fischer
Institute of Mathematics
University of Wuerzburg
Emil-Fischer-Str. 30
97074 Wuerzburg
GERMANY

Phone: +49 931 3188911
E-mail: tom.fischer@uni-wuerzburg.de